

NUMERICAL WEIL-PETERSSON METRICS ON MODULI SPACES OF CALABI-YAU MANIFOLDS

JULIEN KELLER AND SERGIO LUKIC

ABSTRACT. We introduce a simple and very fast algorithm that computes Weil-Petersson metrics on moduli spaces of polarized Calabi-Yau manifolds. Also, by using Donaldson's quantization link between the infinite and finite dimensional G.I.T quotients that describe moduli spaces of varieties, we define a natural sequence of Kähler metrics. We prove that the sequence converges to the Weil-Petersson metric. We also develop an algorithm that numerically approximates such metrics, and hence the Weil-Petersson metric itself. Explicit examples are provided on a family of Calabi-Yau Quintic hypersurfaces in \mathbb{CP}^4 . The scope of our second algorithm is much broader; the same techniques can be used to approximate metrics on null spaces of Dirac operators coupled to Hermite Yang-Mills connections.

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1. INTRODUCTION

Research on differential geometry of complex manifolds has reached some impressive results on the existence of solutions to difficult non-linear PDE's. Yau's proof of Calabi's conjecture [Yau] and Donaldson-Uhlenbeck-Yau's proof of the existence of Hermite-Einstein metrics on stable holomorphic vector bundles [Don1, UY], are main examples of such theorems. Although only very rarely does one expect to find explicit formulae for the solutions, one can explore the geometry of the solutions using numerical methods. During the last years, several techniques that approximate Kähler-Einstein and Hermite-Einstein metrics have appeared in the literature, mainly due to [Don6].

In this paper we begin numerical work to study Weil-Petersson metrics on moduli spaces of such solutions. The main focus is on moduli spaces of complex structures on polarized Calabi-Yau manifolds. First, we develop a fast algorithm that computes the metric by evaluating numerically the exact Weil-Petersson formula. Secondly, by using Donaldson's quantization link between infinite and finite G.I.T quotients, we introduce a sequence of Kähler metrics. We prove that the sequence converges to the Weil-Petersson metric on moduli of manifolds that carry Kähler metrics with constant scalar curvature (cscK metrics in short). Finally, we introduce an algorithm that computes such metrics and discuss an example. We hope this work illuminates the techniques and difficulties that appear when approximating Weil-Petersson metrics on more general moduli spaces.

Motivation for this work can be found in different sources. For instance, we find specially motivating the program by Douglas et. al. [DKLR], building on the work by Donaldson [Don6], to numerically compute Kähler metrics that appear in Calabi-Yau compactifications of string theory. Other source of motivation comes from the study of global Weil-Petersson geometry on moduli spaces of Calabi-Yau manifolds, [DL]. In this case, one of the algorithms introduced in this paper should allow to estimate Weil-Petersson volumes of moduli spaces in a sensible amount of time and with reasonable precision.

1.1. Motivation. Many approaches to unify particle physics attempt to describe known physics by considering a simple field theory defined on a higher dimensional space, and taking four-dimensional limits. The idea, today known as compactification of a field theory, has inspired much work in the interface between geometry and physics. Determining the action functional for fields, in four-dimensional limits, and for a large family of compactifications, is the main mathematical motivation for this work.

Remark 1. For the purpose of this introduction, by a *field theory* we mean a functional space \mathcal{M} of geometric data on a manifold Y (such as Riemannian metrics, connections on a principal bundle on Y , sections of vector bundles, ...), with an action functional $S: \mathcal{M} \rightarrow \mathbb{R}$ defined on it.

A compactification is then a field theory on a D -dimensional space-time which is the product of the 4-dimensional space-time \mathbb{R}^4 with a m -dimensional manifold X , the compactification manifold, carrying a Riemannian metric and other geometric structure corresponding to other fields in the theory. These must solve the Euler-Lagrange equations associated to S , and preserve four dimensional Poincaré

invariance. The most general metric ansatz for a Poincaré invariant compactification is

$$g_{IJ} = \begin{pmatrix} f\eta_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}$$

where the tangent space indices are $0 \leq I < 4 + m = D$, $0 \leq \mu < 4$, and $1 \leq i \leq m$. Here $\eta_{\mu\nu}$ is the Minkowski metric, g_{ij} is a metric on X , and f is a real valued function on X . As the simplest example, consider the D -dimensional Hilbert-Einstein action for general relativity. In this case, Einstein's equations reduce to Ricci flatness of g_{IJ} . Given our metric ansatz, this requires f to be constant, and the metric g_{ij} on X to be Ricci flat.

Typically, if a manifold admits a Ricci flat metric, it will not be unique; rather there will be a moduli space of such metrics. Physically, one then expects to find solutions in which the choice of Ricci flat metric on X is slowly varying in four dimensional space-time. General arguments imply that such variations must be described by variations of 4-dimensional fields, governed by an EFT. For simplicity, by this *Effective Field Theory* (EFT) we mean a four dimensional field theory that emerges in the small radius limit of X , when the geometric data on $\mathbb{R}^4 \times X$ restricted to X satisfies the Euler Lagrange equations. Thus, the action functional of the EFT is defined on a functional space of geometric data on \mathbb{R}^4 .

Given an explicit parametrization of the family of metrics, say $g_{ij}(t_a)$ for some parameters t_a , the EFT could be computed explicitly by promoting the parameters $\{t_a\}$ to 4-dimensional fields $\{t_a(x)\}$, substituting this parametrization into the D -dimensional action, and expanding in powers of the 4-dimensional derivatives. For the Hilbert-Einstein action, we find the four-dimensional effective action functional

$$\begin{aligned} S_{EFT}^{GR} &= \int_{\mathbb{R}^4 \times X} d^{(10)} \text{Vol} \, \text{scal}(g_{IJ}) \\ &= \int_{\mathbb{R}^4 \times X} d^4 x d^m y \sqrt{\det g(t)} \text{scal}(g_{ij}) + \\ (1.1) \quad &\int_X d^m y \sqrt{\det g(t)} g^{ik}(t) g^{jl}(t) \frac{\partial g_{ij}}{\partial t_a} \frac{\partial g_{kl}}{\partial t_b} \times \int_{\mathbb{R}^4} d^4 x \partial_\mu t_a(x) \partial^\mu t_b(x) + \dots \end{aligned}$$

where y^i denotes a local coordinate chart on X , x^μ a local coordinate chart on \mathbb{R}^4 , and $\text{scal}(g)$ is the scalar curvature associated to the D dimensional metric. In general, a direct computation of (1.1) is impossible. This becomes especially clear when one learns that the Ricci flat metrics g_{ij} are not explicitly known for the examples of interest.

An interesting class of compactifications come from the field theory limit of string theories, where the space-time dimension is $D = 10$. Requiring $\mathcal{N} = 1$ supersymmetry on the four dimensional EFT and the vanishing of torsion elements, fixes X to be a Calabi-Yau threefold. In this case, computing the four dimensional action functional for the $\{t_a(x)\}$ fields (1.1) involves to know the Weil-Petersson metric on the moduli space of Kähler Ricci flat metrics on X .

These theories contain other objects besides the space-time metric. For instance, in a heterotic string theory [GSW], the geometric content also involves a principal $E_8 \times E_8$ -bundle endowed with a gauge connection A ; E_8 denotes the Cartan's exceptional simple Lie group of dimension 248. In a Poincaré invariant compactification, one defines the theory on a principal $E_8 \times E_8$ -bundle $P \rightarrow \mathbb{R}^4 \times X$. For every

point x on X , the restriction of the principal bundle P to $\mathbb{R}^4 \times x$ is trivial, i.e. $P|_{\mathbb{R}^4 \times x \hookrightarrow \mathbb{R}^4 \times X}$ is equivalent to $E_8 \times \mathbb{R}^4$.

In the small radius limit of X one obtains an effective gauge theory on \mathbb{R}^4 with gauge group H , by expanding the Yang-Mills functional around a background reducible connection A_0 on $P \rightarrow \mathbb{R}^4 \times X$. For simplicity, one considers a subgroup G of E_8 and takes A_0 to be a connection on a principal G -subbundle of $P \rightarrow \mathbb{R}^4 \times X$. The gauge group H of the effective theory on \mathbb{R}^4 is the commutant of $G \hookrightarrow E_8$. In many applications G is the special unitary group $SU(r)$, with $2 < r < 6$. The Euler-Lagrange equations associated to the Yang-Mills functional require A_0 to be a Hermite Yang-Mills unitary connection.

As in the case of the Kähler Ricci flat metric, if the bundle admits a Hermite Yang-Mills connection, it will not be unique; rather there will be a moduli space of E_8 connections on P with G -holonomy. Although a general description of such moduli spaces is not explicitly known for examples of interest, it is interesting enough to work with the space of local deformations around a particular A_0 . Such space is in one to one correspondence with the null space of the Dirac operator on X , coupled to A_0 , that acts on spinors which are sections of an associated vector bundle to P , adjoint representation of E_8 . The decomposition of such null space as the direct sum of irreducible representations of H , corresponds to the particle spectrum in the four dimensional effective theory (see Table 1).

$G \times H$	Particle Spectrum from the breaking of 248
$SU(3) \times E_6$	$V_{(1,78)} = E_6$ Gauge Field $V_{(3,27)} = H^1(X, E)$ $V_{(\bar{3},27)} = H^1(X, E^*) = H^2(X, E)$ $V_{(8,1)} = H^1(X, E \otimes E^*)$
$SU(4) \times SO(10)$	$V_{(1,45)} = SO(10)$ Gauge Field $V_{(4,16)} = H^1(X, E)$ $V_{(\bar{4},16)} = H^1(X, E^*) = H^2(X, E)$ $V_{(6,10)} = H^1(X, \wedge^2 E)$ $V_{(15,1)} = H^1(X, E \otimes E^*)$
$SU(5) \times SU(5)$	$V_{(1,24)} = SU(5)$ Gauge Field $V_{(5,10)} = H^1(X, E)$ $V_{(\bar{5},10)} = H^1(X, E^*) = H^2(E)$ $V_{(10,5)} = H^1(X, \wedge^2 E^*)$ $V_{(\bar{10},5)} = H^1(X, \wedge^2 E)$ $V_{(24,1)} = H^1(X, E \otimes E^*)$

TABLE 1. A vector bundle E with structure group G can break the E_8 gauge group of the heterotic string into Great Unification Theory groups H , such as E_6 , $SO(10)$ and $SU(5)$. The low-energy representations under $\mathfrak{G} \times \mathfrak{H}$ are found from the branching of the adjoint representation of E_8 denoted by **248**. The representations of $\mathfrak{G} \times \mathfrak{H}$ are denoted by their respective dimensions $V_{(\dim R_{\mathfrak{G}}, \dim R_{\mathfrak{H}})} = R_{\mathfrak{G}} \otimes R_{\mathfrak{H}}$. The particle spectrum is obtained by computing the indicated sheaf cohomology groups.

Thus, in order to find the action functional that governs the dynamics of such particles on \mathbb{R}^4 , one has to expand the 10-dimensional action functional in the small radius limit of X , for small perturbations of A_0 that preserve the linearized Yang-Mills equations on $P \rightarrow X$, and Poincaré invariance on \mathbb{R}^4 .

More precisely, given a local coordinate chart $\{z^i, \bar{z}^{\bar{j}}\}_{i,j=1}^3$ on X , $\{x^\mu\}_{\mu=1}^4$ on \mathbb{R}^4 , and a trivialization of P one can expand the gauge connection A around A_0 as

$$A(z, x) = A_{0,i} dz^i + A_{0,\bar{j}} d\bar{z}^{\bar{j}} + A_\mu(x) dx^\mu + t_p^*(x) \frac{\partial A_{\bar{j}}}{\partial t_p} d\bar{z}^{\bar{j}} + t_p(x) \frac{\partial A_i}{\partial t_p} dz^i + \dots$$

Here, $A_\mu dx^\mu$ is the 4-dimensional H -gauge connection and $\{t_p\}$ is a local coordinate chart on the space of infinitesimal deformations of the connection A_0 that preserve the linearized Yang-Mills equations. The $\{t_p\}$ are parameters since the point of view of the Calabi-Yau geometry; though, they are four dimensional charged scalar fields $\{t_p(x)\}$ since the point of view of 4-dimensional field theory. By the ellipsis, we denote higher order corrections in t and, also, corrections by terms which do not preserve the linearized Yang-Mills equations; one can assume that both corrections are irrelevant in low energy physics. If we expand the pure Yang-Mills action in 10 dimensions assuming our Poincaré invariant ansatz, we find

$$\begin{aligned} S_{EFT}^{YM} &= \int_{\mathbb{R}^4 \times X} d^{(10)} \text{Vol} \text{Tr} (F_{IJ} F^{IJ}) = \int_{\mathbb{R}^4 \times X} d^{(10)} \text{Vol} \text{Tr} (F_{i\mu} F_{\bar{j}\nu}) g^{i\bar{j}} \eta^{\mu\nu} \dots = \\ (1.2) \quad &= \int_X d^{(6)} \text{Vol} \text{Tr} \left(\frac{\partial A_i}{\partial t_p} \frac{\partial A_{\bar{j}}}{\partial t_p} \right) g^{i\bar{j}} \times \int_{\mathbb{R}^4} d^4 x \partial_\mu t_p \partial^\mu t_q^* + \dots \end{aligned}$$

Here, we are using the usual Einstein's conventions for summation.

Hence, an understanding of the effective action for the t fields, known as charged matter and eventually related to particles such as electrons, quarks, etc., requires to compute generalized Weil-Petersson metrics (as in (1.2)) on the moduli space of E_8 connections on P with G -holonomy. Proper generalizations of the numerical tools that we introduce in this paper, should be useful in the case when $G = SU(r)$ and the principal $SU(r)$ -subbundle underlies a family of stable holomorphic vector bundles $E \rightarrow X$ (with $c_1(E) = 0$, $\text{rank}(E) = r$). In this case one can use balanced embeddings to approximate the Hermite Yang-Mills connections, identify the space of infinitesimal deformations of the background connection with sheaf cohomology groups (see Table 1), and approximate the Weil-Petersson metrics by using the metrics that we define in this paper.

1.2. Outline of the paper. This paper is organized as follows. In section 2 we review some general results on moduli spaces of polarized Calabi-Yau manifolds, and define their corresponding Weil-Petersson metrics. We explain our first method to numerically compute the Weil-Petersson metric in section 3. By combining formulae of deformations of the holomorphic top form under infinitesimal diffeomorphisms, and Monte Carlo integration techniques, we evaluate the Weil-Petersson metric in a particular example. Section 4 is the core of the paper. First, we review basic concepts on moduli spaces of polarized varieties since the point of view of Geometric Invariant Theory. After defining a sequence of Kähler metrics, we prove that it converges to the Weil-Petersson metric on the moduli space of constant scalar curvature Kähler metrics (Kähler Ricci flat is a particular case for Calabi-Yau varieties). Also, we outline how a similar idea should work for Weil-Petersson

metrics on moduli spaces of vector bundles. We conclude by computing explicit examples on the Quintic threefold in \mathbb{P}^4 .

1.3. Notation. Throughout this paper X denotes a smooth projective Calabi-Yau manifold of complex dimension n . Let ν , with $\text{span}(\nu) = H^0(X, \mathcal{K}_X)$, be the corresponding holomorphic n -form, and \mathcal{L} the defining polarization, i.e. an ample line bundle on X . We denote by ω the Kähler two form, with $[\omega] = c_1(\mathcal{L})$. By $g_{i\bar{j}}$ we mean the compatible Riemannian metric on X , and by h the compatible Hermitian metric on \mathcal{L} whose curvature is $c_1(h) = \omega$.

2. WEIL-PETERSSON METRICS ON MODULI OF POLARIZED MANIFOLDS

A *holomorphic family* of compact polarized Kähler manifolds (X_t, g_t) parametrized by $t \in \mathcal{T}$ is a complex manifold \mathcal{X} together with a proper holomorphic map $\pi: \mathcal{X} \rightarrow \mathcal{T}$ which is of maximal rank. This means that the differential of π is surjective everywhere, and that $\pi^{-1}(t)$ is compact for any $t \in \mathcal{T}$.

Given a base point $0 \in \mathcal{T}$ we say that $\pi^{-1}(t) = X_t$ is a deformation of X_0 . Locally, \mathcal{X} is a trivial fiber product $\mathcal{X}|_{\mathcal{U}} \simeq \mathcal{U} \times X_t$. If $T_t\mathcal{T}$ denotes the holomorphic tangent space to \mathcal{T} at t , we can define the infinitesimal deformation or Kodaira-Spencer map:

$$\rho_t: T_t\mathcal{T} \longrightarrow H^1(X_t, TX_t).$$

where $H^1(X_t, TX_t)$ can be identified with the harmonic representatives of $(0,1)$ forms with values in the holomorphic tangent bundle $TX_t = T^{1,0}X_t$; in other words $H^1(X_t, TX_t) \sim H_{\bar{\partial}}^{0,1}(TX_t)$. We know that the Kähler metric g_t induces a metric on $\Lambda^{0,1}(TX_t)$. Thus, for $v_1, v_2 \in T_t\mathcal{T}$, we can define a Kähler metric at $t \in \mathcal{T}$,

$$(2.1) \quad G(v_1, v_2) = \int_{X_t} \langle \rho(v_1), \rho(v_2) \rangle_{g(t)} d\text{Vol}(g(t)).$$

Note that G is possibly degenerate. If ρ is injective and $g(s)$ satisfy an Einstein type condition, one says that G is a Weil-Petersson metric on the Kuranishi space.

2.1. Weil-Petersson metric for Calabi-Yau's manifolds. Suppose now that $\mathcal{X} \rightarrow \mathcal{T}$ is a family of polarized Calabi-Yau manifolds (X, \mathcal{L}) , naturally equipped with a unique Ricci-flat Kähler metric in a given Kähler class. We can identify the tangent space at $t \in \mathcal{T}$, $T_t\mathcal{T}$ with $H_{\bar{\partial}}^{0,1}(TX_t)$. This allows us to define the Weil-Petersson metric on \mathcal{T} , the local moduli space of (X, \mathcal{L}) , as follows.

Definition 1. Let $v_1, v_2 \in T_t\mathcal{T} \simeq H_{\bar{\partial}}^{0,1}(TX_t)$, then

$$\langle v_1, v_2 \rangle_{W.P.} := \int_{X_t} v_{1\bar{k}}^i \overline{v_{2\bar{l}}^j} g_{i\bar{j}} g^{l\bar{k}} d\text{Vol}.$$

In this particular case Tian and Todorov proved the following

Theorem 1. (Tian-Todorov, [Tia, Tod]) *Let $\pi: \mathcal{X} \rightarrow \mathcal{T} \ni 0$, $\pi^{-1}(0) = X_0$, be the family of X , then \mathcal{T} is a non-singular complex analytic space such that*

$$\dim_{\mathbb{C}} \mathcal{T} = \dim_{\mathbb{C}} H^1(X_t, TX_t) = \dim_{\mathbb{C}} H^1(X_t, \Omega^{n-1}),$$

where TX_t denotes the sheaf of holomorphic vector fields on X_t , and Ω^{n-1} the sheaf of holomorphic $(n-1)$ forms.

There is a correspondence between $H_{\bar{\partial}}^{0,1}(TX_t)$ and $H^1(X_t, \Omega^{n-1})$ given by the interior product and the global holomorphic n -form on X . Then, one can evaluate the Weil-Petersson metric in terms of the standard cup product on $H^{n-1,1}(X_t)$. Indeed,

$$(2.2) \quad \Psi(t, \bar{t}) = (-1)^{\frac{n(n-1)}{2}} i^{n-2} \log \left(\int_X \nu_t \wedge \bar{\nu}_t \right)$$

is the local Kähler potential for the Weil-Petersson metric. This is an important formula; for instance, if we fix the differential structure on X and consider variations of the complex structure in the holomorphic top form ν_t , one can evaluate $\partial\bar{\partial}\Psi$ by computing differentials $\frac{\partial\nu_t}{\partial t_a}$, with $\frac{\partial}{\partial t_a}$ a basis for $T_t\mathcal{T} = H^1(X_t, \Omega^{n-1})$. This idea will play an important role in the next section, where we will perform a direct calculation of $\partial\bar{\partial}\Psi$. Also, one could compute $\partial\bar{\partial}\Psi$ using the standard cup product to be able to express $\int_X \nu_t \wedge \bar{\nu}_t$ as a function of t , as we show in the following example.

2.2. Example: the Quintic in \mathbb{P}^4 . In this paper we will study different constructions on the Quintic hypersurface $X = Q$ in \mathbb{P}^4 , with $h^{1,1} = 1$ and $\dim_{\mathbb{C}} H^1(Q, \Omega^2) = h^{2,1} = 101$. Many geometrical properties of this Calabi-Yau 3-fold are known in the literature. For instance, one can describe its moduli very explicitly. If we define

$$W = \{ P \mid P \text{ a homogeneous quintic polynomial of } Z_0, Z_1, Z_2, Z_3, Z_4 \},$$

one can verify that $\dim W = 126$. Hence, for any $t \in \mathbb{P}W = \mathbb{P}^{125}$, t is represented by a hypersurface in \mathbb{P}^4 . As two hypersurfaces that differ by an element in $\text{Aut}(\mathbb{P}^4)$ are equivalent, and there exists a divisor \mathcal{D} in \mathbb{P}^{125} characterizing the singular hypersurfaces in \mathbb{P}^4 , the moduli space of Quintics Q is given by

$$\mathcal{M} = (\mathbb{P}^{125} \setminus \mathcal{D}) / \text{Aut}(\mathbb{P}^4).$$

The dimension of the moduli space is 101, as expected.

For simplicity, in this paper we study a one dimensional subspace of complex deformations defined by

$$P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4,$$

and parametrized by t . As t and $t \exp(2\sqrt{-1}\pi l/5)$, for any $l \in \mathbb{Z}$, represent the same variety, the fundamental region on the t -plane is defined as $\{ t \mid 0 \leq \arg(t) < 2\pi/5 \text{ and } t \neq 1 \}$. For t a fifth root of unity, i.e. $t = \exp(2\sqrt{-1}\pi l/5)$ for any $l \in \mathbb{Z}$, the Quintic develops double point singularities.

2.2.1. Evaluating the Weil-Petersson metric on the family of Quintics. Candelas, de la Ossa, Green and Parkes [COGP] evaluated the volume $\int_X \nu_t \wedge \bar{\nu}_t$ as function of t , for the family of Quintic 3-folds

$$(2.3) \quad P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4,$$

by evaluating cup products. More specifically, they constructed explicitly a symplectic basis of 3-cycles (A^a, B_b) for $H_3(Q, \mathbb{Z})$, such that

$$A^a \cap B_b = \delta_b^a, \quad A^a \cap A^b = 0, \quad B_a \cap B_b = 0.$$

Also, they considered the dual basis (α_a, β^b) in cohomology so that

$$\int_{A^a} \alpha_b = \delta_b^a, \quad \int_{B_a} \beta^b = \delta_a^b,$$

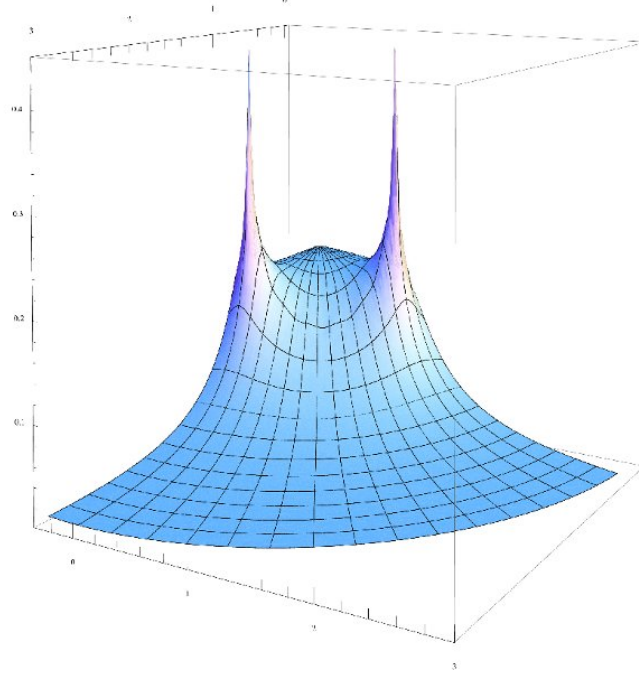


FIGURE 1. Weil-Petersson metric (vertical axis) on the t -plane (horizontal plane) of Calabi-Yau Quintic 3-folds, $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4$.

with the other integrals vanishing. Then it follows that

$$\int_Q \alpha_a \wedge \beta^b = \delta_a^b, \quad \int_Q \alpha_a \wedge \alpha_b = \int_Q \beta^a \wedge \beta^b = 0.$$

Thus, the holomorphic three-form ν_t can be expanded using this basis as

$$\nu_t = z^a \alpha_a - \mathcal{G}_b \beta^b,$$

and therefore the volume can be written as

$$\int_Q \nu_t \wedge \bar{\nu}_t = \bar{z}^a \mathcal{G}_a - z^a \bar{\mathcal{G}}_a.$$

Using this, the Weil-Petersson metric is

$$(2.4) \quad g_{t\bar{t}} = -i \partial_t \bar{\partial}_t \log (\bar{z}^a \mathcal{G}_a - z^a \bar{\mathcal{G}}_a).$$

Hence, in order to obtain the Weil-Petersson metric, it is sufficient to evaluate the periods $z^a = \int_{A^a} \nu$, $\mathcal{G}_b = \int_{B_b} \nu$. For that, let us consider the vector space V_j generated by the vectors

$$\begin{pmatrix} \frac{\partial^k}{\partial t^k} z^a(t) \\ \frac{\partial^k}{\partial t^k} \mathcal{G}_b(t) \end{pmatrix}$$

for $0 \leq k \leq j$. For generic values t of the Kuranishi deformation, the dimension of V_j must be constant and in our case, this dimension cannot be larger than 4. Thus, expressing one element of V_5 from the others, we obtain a non-trivial

ordinary differential equation relating the periods. This is the so-called Picard-Fuchs equation, and we refer to [Mor] for a mathematical approach to this topic. Note that the form of these equations depends on the local coordinates over the space of deformations and the choice of the holomorphic form $\nu(t)$. The solution of the Picard-Fuchs equations may be singular but the types of singularities that can occur have been well studied. In [COGP], those equations have generalized hypergeometric type and can be solved by expressing the integrands of the periods in power series of t . Each coefficient of the power series leads to an integral that can be evaluated by residue formulae. The obtained periods are extended by analytic continuation over fundamental domains ($|t| < 1$ with $0 < \arg(t) < \frac{2\pi}{5}$ and $|t| > 1$ with $0 < \arg(t) < \frac{2\pi}{5}$). Although the behavior of the periods can be described at the singular points (in our case $t = 1, \infty$), it is quite difficult to obtain simple formulas to express exactly the periods if one is not considering hypersurfaces. We have written a simple program in Mathematica and Maple, for the case of the family of quintics (2.3), and computed numerically the power series that define the periods. Fig. 1 shows our evaluation of (2.4) for $0 < |t| \leq 3$ and $0 \leq \arg(t) < 2\pi/5$.

3. NUMERICAL EVALUATION VIA DEFORMATIONS OF THE HOLOMORPHIC FORM

In this section we describe how to approximate Weil-Petersson metrics by considering variations of the holomorphic n -form. First, we make the following important distinction: by X we denote a Calabi-Yau differentiable manifold with no complex structure defined on it. X_t denotes the same differentiable manifold endowed with an integrable complex structure parametrized by t . We denote by $U \subset X$ an open subset of the differentiable manifold X , such that $U \subset X$ is independent of any complex structure one defines on X .

Every element in $v \in T_{t_0}\mathcal{T}$ yields an infinitesimal deformation of the complex structure on X_{t_0} . By going to a local coordinate patch on $U \subset X$ we can relate the holomorphic coordinates on X_{t_0} with the holomorphic ones on X_{t_0+tv} by defining a proper infinitesimal diffeomorphism. Let $\{w^i\}_{i=1}^n$ be a local holomorphic coordinate system for X_{t_0} on $U \subset X$, and $\{y^i\}_{i=1}^n$ be a local holomorphic coordinate system for X_{t_0+tv} on the same subset U . Therefore, on U , we can relate the w -coordinates and the y -coordinates as:

$$(3.1) \quad y^i = w^i + v^a \vartheta_a^i(w, \bar{w}) + O(v^2),$$

with ϑ a vector field, non-holomorphic section of $T^{1,0}X_{t_0}$, and $\frac{\partial}{\partial t_a}$ is a basis for $T_{t_0}\mathcal{T}$.

Hence, we can write the holomorphic top form ν_{t_0+tv} on X_{t_0+tv} , using the w -coordinate system, as a non-holomorphic n -form in $\Omega^{n,0}(X_{t_0}) \oplus \Omega^{n-1,1}(X_{t_0})$. More precisely,

$$(3.2) \quad \nu_{t_0+tv} = \nu_{t_0} + v^a \partial_{t_a} \nu_{t_0} + O(v^2),$$

where the $O(v^2)$ terms are irrelevant for the purpose of this paper. The term $\partial_{t_a} \nu_{t_0}$ is computed as pull-back of the infinitesimal diffeomorphism defined by $\vartheta_a^i(w, \bar{w})$. Thus, given a basis of deformations $\partial_{t_a} \nu_{t_0} \in \Omega^{n,0}(X_{t_0}) \oplus \Omega^{n-1,1}(X_{t_0})$ and vectors $v_1, v_2 \in T_{t_0}\mathcal{T}$, we can write the Weil-Petersson inner product as

$$(3.3) \quad \langle v_1, v_2 \rangle_{W.P.} = -\frac{v_1^a \bar{v}_2^b \int_X \partial_{t_a} \nu_{t_0} \wedge \overline{\partial_{t_b} \nu_{t_0}}}{\int_X \nu_{t_0} \wedge \overline{\nu_{t_0}}} + \frac{v_1^a \bar{v}_2^b \int_X \partial_{t_a} \nu_{t_0} \wedge \overline{\nu_{t_0}} \int_X \nu_{t_0} \wedge \overline{\partial_{t_b} \nu_{t_0}}}{\left(\int_X \nu_{t_0} \wedge \overline{\nu_{t_0}} \right)^2},$$

where we have expanded the Kähler potential (2.2) for n -forms as (3.2). Therefore, a direct calculation of the Weil-Petersson metric involves:

- A choice of $\vartheta_a^i(w, \bar{w})$, which is not unique and depends on the particular geometry of the Calabi-Yau manifold.
- To perform several integrals on X .

If X_{t_0} is a Complete Intersection Calabi Yau manifold there is a natural choice for $\vartheta_a^i(w, \bar{w})$. Let $\{P_\alpha(Z)_t\}_{\alpha=1}^{m-n}$ be a family of homogeneous polynomials in \mathbb{P}^m whose common zero loci define X_t . Let us suppose that at $t = t_0$, given two independent deformations of the complex structure, $v_1, v_2 \in T_{t_0}\mathcal{T} \simeq H_{\bar{\partial}}^{0,1}(TX_{t_0})$, we can find two sets of polynomials, $\{\delta_1 P_\alpha(Z) = v_1^a \partial_{t_a} P_\alpha(Z)_t\}_{\alpha=1}^{m-n}$ and $\{\delta_2 P_\alpha(Z) = v_2^a \partial_{t_a} P_\alpha(Z)_t\}_{\alpha=1}^{m-n}$, that parametrize isomorphic deformations of the complex structure. We set a coordinate atlas on $X_{t_0} \subset \mathbb{P}^m$ by choosing inhomogeneous local coordinates $\{w_i = Z_i/Z_0\}_{i=1}^m$ on \mathbb{P}^m , n coordinates as local coordinates on $U \subset X_{t_0}$, and the remaining $n - m$ coordinates as dependent of the n coordinates on $U \subset X_{t_0} \subset \mathbb{P}^m$. In other words, for any point $x \in X_{t_0}$, by making a unitary change of coordinates on \mathbb{P}^m we can always set $\{w_i\}_{i=1}^n$ to be a local coordinate system on an open subset of X_{t_0} that contains x , while the remaining coordinates $\{w_i = w_i(w_1, \dots, x_n)\}_{i=n+1}^m$ on $X_{t_0} \subset \mathbb{P}^m$ can be expressed as a function of $\{w_i\}_{i=1}^n$. We write the defining polynomials in inhomogeneous coordinates, as

$$\begin{aligned} p_\alpha(w) = p_\alpha(w)_t &= P_\alpha(Z)_t / Z_0^{\deg P_\alpha}, \\ \partial_{t_a} p_\alpha(w) &= \partial_{t_a} (P_\alpha(Z)_t / Z_0^{\deg P_\alpha}), \end{aligned}$$

where $\deg P \in \mathbb{Z}^+$ is the degree of the homogeneous polynomial P . If $\vartheta_a^i(w, \bar{w})$ are vector fields on $X_{t_0} \subset \mathbb{P}^m$ corresponding to the deformations $\{\partial_{t_a} p_\alpha(w)\}$, and $\{y_i\}_{i=1}^m$ is a holomorphic local coordinate system on $X_{t_0+v^a} \subset \mathbb{P}^m$, the following equation holds for an infinitesimal variation v^a on the moduli,

$$(3.4) \quad p_\alpha(y) + v^a \partial_{t_a} p_\alpha(y) = 0 = p_\alpha(w) + v^a \frac{\partial p_\alpha(w)}{\partial w_i} \vartheta_a^i(w, \bar{w}) + v^a \partial_{t_a} p_\alpha(w) + O(v^2),$$

where the repeated index a is not summed this time, and y^i obeys (3.1).

Proposition 1. *Let $G_{i\bar{j}}$ be a Fubini-Study metric on \mathbb{P}^m . Let $H_{\alpha\bar{\beta}}$ be the elements*

$$H_{\alpha\bar{\beta}} = G^{i\bar{j}} \frac{\partial p_\alpha(w)}{\partial w_i} \frac{\partial \bar{p}_\beta(\bar{w})}{\partial \bar{w}_{\bar{j}}}.$$

Then, a natural choice for $\vartheta_a^i(w, \bar{w})$ is

$$(3.5) \quad \vartheta_a^i(w, \bar{w}) = - (H^{-1})^{\bar{\beta}\gamma} G^{i\bar{j}} \frac{\partial \bar{p}_\beta(\bar{w})}{\partial \bar{w}_{\bar{j}}} \partial_{t_a} p_\gamma(w).$$

The proof is straightforward by substituting $\vartheta_a^i(w, \bar{w})$ into the equation (3.4), and observing that $p_\alpha(w) = 0$, as w lies on $X_{t_0} \subset \mathbb{P}^m$.

We can calculate the deformation of ν_t under the infinitesimal diffeomorphism defined by (3.1), by combining (3.5) and (3.2). More precisely, if

$$(3.6) \quad \nu_{t_0+v^a} = N_{i_1, \dots, i_n}(y) dy^{i_1} \wedge \dots \wedge dy^{i_n},$$

is the holomorphic n -form on $X_{t_0+v^a} \subset \mathbb{P}^m$, $y^i = w^i + v^a \vartheta_a^i(w, \bar{w}) + O(v^2)$, and

$$(3.7) \quad dy^i = dw^i + v^a \frac{\partial \vartheta_a^i(w, \bar{w})}{\partial w^j} dw^j + v^a \frac{\partial \vartheta_a^i(w, \bar{w})}{\partial \bar{w}^{\bar{j}}} d\bar{w}^{\bar{j}} + O(v^2),$$

we can expand (3.6) as in (3.2), and determine the term $\frac{\partial \nu_t}{\partial t_a}(t_0)$ that we need to evaluate the Weil-Petersson metric (3.3).

Although one could compute (3.2) in the general case, for simplicity we just consider the case $m - n = 1$, i.e. X_{t_0} is a hypersurface defined as the zero locus of a polynomial $p(w)$. By the adjunction formula we know that $\nu_{t_0+v^a}$ is pull-back of a meromorphic n -form on \mathbb{P}^{n+1} that obeys the simple formula

$$(3.8) \quad \prod_{i=1}^{n+1} dy^i = d(p(y) + v^a \partial_{t_a} p(y)) \wedge \nu_{t_0+v^a}.$$

Using $\nu_{t_0+v^a} = \nu_{t_0} + v^a \partial_{t_a} \nu_{t_0} + O(v^2)$ and the transformation of dy^i (3.7), in (3.8), one can compute $\partial_{t_a} \nu_{t_0}$ as

$$(3.9) \quad \partial_{t_a} \nu_{t_0} = -\frac{1}{\frac{\partial p}{\partial w^{n+1}}} \left(\sum_{i=1}^{n+1} \frac{\partial \vartheta_a^i(w, \bar{w})}{\partial w^i} \right) \prod_{i=1}^n dw^i - \frac{1}{\frac{\partial p}{\partial w^{n+1}}} \times \\ \left(\sum_{i,j=1}^n (-1)^{n-i} \left(\frac{\partial \vartheta_a^i(w, \bar{w})}{\partial \bar{w}^j} + \frac{\partial \bar{w}^{n+1}}{\partial \bar{w}^j} \frac{\partial \vartheta_a^i(w, \bar{w})}{\partial \bar{w}^{n+1}} \right) dw^1 \dots \widehat{dw^j} \dots dw^n d\bar{w}^j \right) + \dots$$

Here, the differentials dw^i obey the Grassmann algebra of forms, $\widehat{dw^i}$ denotes the omission of dw^i , and the final ellipsis denotes further terms which do not contribute to the equation (3.3). Now, equipped with a local formula for the integrands of equation (3.3), we need a numerical way to evaluate the integrals that appear therein.

3.1. Monte Carlo integration on varieties. One of the problems that we need to solve, in order to compute the Weil-Petersson metric via local deformations of the holomorphic top-form, consists in evaluating integrals of the type

$$(3.10) \quad \int_X f \nu \wedge \bar{\nu}.$$

We can numerically approximate such integrals by introducing an auxiliary measure $d\mu$, and generating random points $\{q_l \in X\}_{1 \leq l \leq NP}$ on X uniformly distributed under $d\mu$. Hence, by defining the mass function $m(x) = \nu \wedge \bar{\nu} / d\mu(x)$, we can estimate (3.10), *a la* Monte Carlo, as

$$(3.11) \quad \int_X f \nu \wedge \bar{\nu} \simeq \frac{\text{vol}(X)}{\sum_l m_l} \sum_{l=1}^{NP} f(q_l) m(q_l) + O\left(\frac{1}{\sqrt{NP}}\right),$$

where $NP \in \mathbb{Z}^+$ is the number of points used and $O(NP^{-1/2})$ is the standard error for large NP ; which is proportional to $\frac{1}{\sqrt{NP}}$ by the central limit theorem.

In the particular case of a polarized manifold with a very ample line bundle \mathcal{L} , we generate the point set and the auxiliary measure $d\mu$, by taking projective embeddings $\iota: X \hookrightarrow \mathbb{P}H^0(X, \mathcal{L})^*$. We endow such projective space with a Fubini-Study metric ω_{FS} and consider random sections σ in $\mathbb{P}H^0(X, \mathcal{L})$ with respect to the volume form associated to the Fubini-Study metric. The zero locus of such random sections σ are divisors with associated zero currents T_σ . One can show [SZ] that the expected zero current is:

$$E(T_\sigma) = \iota^* \omega_{FS}.$$

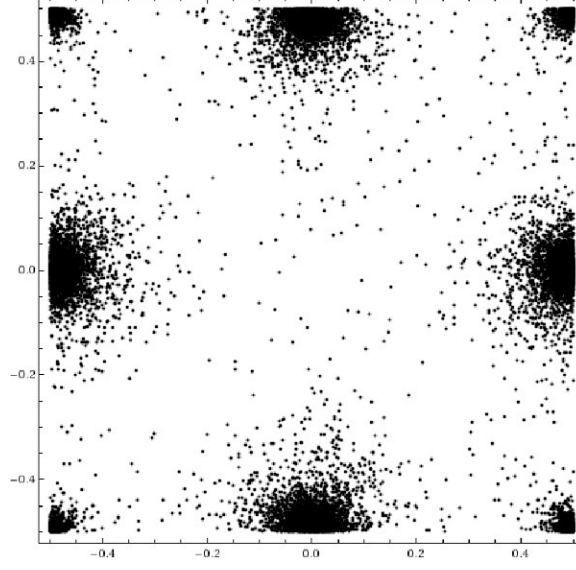


FIGURE 2. Distribution of random points on the Weierstrass cubic $Z_2^2 Z_0 = 4Z_1^3 - 60G_4(i)Z_1 Z_0^2$, under the Fubini-Study metric defined by the Kähler potential $\log(1 + |Z_1/Z_0|^2 + |Z_2/Z_0|^2)$.

Therefore, the expected zero loci of n independent random sections in $\mathbb{P}H^0(X, \mathcal{L})^*$ are $\int_X c_1(\mathcal{L})^n$ points on X uniformly distributed under

$$E(T_{\sigma_1 \dots \sigma_n}) = \frac{(i^* \omega_{FS})^n}{n!}.$$

Thus, we take $d\mu = i^* \omega_{FS}^n / n!$ as the auxiliary measure and generate the points, uniformly distributed under $d\mu$, by taking the common zero loci of n independent random sections. The mass function is

$$(3.12) \quad m(x) = n! \frac{\nu \wedge \bar{\nu}}{(i^* \omega_{FS})^n}(x).$$

Example

For instance, we consider the elliptic curve E in \mathbb{CP}^2 defined as the zero locus of the Weierstrass cubic polynomial

$$Z_2^2 Z_0 = 4Z_1^3 - 60G_4(i)Z_1 Z_0^2,$$

where $G_4(i)$, the Eisenstein series of index 4 evaluated at the complex parameter $\tau = i$, is $G_4(i) = -3.151212\dots$. This elliptic curve can be seen as the square torus \mathbb{C}/\mathbb{Z}^2 embedded in \mathbb{CP}^2 . The Calabi-Yau area form corresponds to the flat area form inherited from the complex plane \mathbb{C} , in the quotient \mathbb{C}/\mathbb{Z}^2 . Intersections of three random sections in $\mathbb{CP}^2 = \mathbb{P}H^0(E, \mathcal{O}(3pts))$ are equivalent to intersections of the cubic $E \hookrightarrow \mathbb{CP}^2$ with random projective lines $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$. The Fubini-Study area form yields a particular distribution of points as shown in Fig. 2. We can still perform integrals with respect to $\nu \wedge \bar{\nu}$, because we have a precise formula for $\nu \wedge \bar{\nu} / i^* \omega_{FS}$.

3.1.1. *Refinements.* When the ratio of the maximum over the minimum of the mass function (3.12), is very large, one expects a bad behaviour of the Monte Carlo method just described. In this case, one can increase the number of points to try to approximate the integrals with more accuracy and precision. Also, one can use an optimal combination of several Fubini-Study metrics and subsets on X , to generate an Improved Point Set.

Remark 2. By an *Improved Point Set*, we mean a distribution of points on X whose associated mass formula $m_{IPS}(x)$ obeys

$$\frac{\max(m_{IPS}(x))}{\min(m_{IPS}(x))} \ll \frac{\max(m(x))}{\min(m(x))}.$$

For most applications, the most efficient strategy consists in working with a unique optimal Fubini-Study metric, and a point set generated by the intersection of independent random sections under the associated measure. The optimal Fubini-Study metric could be the ν -balanced metric [Don6] (see Definition 5 in section below), which is an accurate approximation to the Kähler Ricci flat metric. The number of points should be adjusted to obtain the required accuracy and precision.

However, if we integrate functions whose evaluation maps are extremely slow since a numerical point of view, we won't be able to use a high number of points to reduce the error. In this case, we should improve the distribution of the point set, while using a constant number of points. The most obvious strategy consists in choosing several Fubini-Study metrics $\{\omega_{FS}^q\}_{q=1}^Q$, and use the mass function

$$(3.13) \quad m_{IPS}(x) = n! \frac{\nu \wedge \bar{\nu}}{(i^* \omega_{FS}^q)^n}(x), \{q \in [1, Q] : \left| n! \frac{\nu \wedge \bar{\nu}}{(i^* \omega_{FS}^q)^n}(x) - 1 \right| \text{ is minimum} \}.$$

Here, we work with normalized volumes, $n! \int_X \nu \wedge \bar{\nu} = \int_X (i^* \omega_{FS}^q)^n$. The mass function (3.13), implies that we can decompose X as a disjoint union of open subsets $X = \coprod_{q=1}^Q U_q$ with non-zero volume. In other words, each Fubini-Study metric $\omega_{FS}^q \in \{\omega_{FS}^q\}_{q=1}^Q$, defines a subset $U_r \subset X$:

$$x \in U_r \text{ if } m_{IPS}(x) = n! \frac{\nu \wedge \bar{\nu}}{(i^* \omega_{FS}^r)^n}(x).$$

Therefore, in this second strategy, sets of n independent random sections with respect to the Fubini-Study volume form $(\omega_{FS}^q)^n$ yield random points on X ; however, only those points that lie on $U_q \subset X$ are accepted. If a point $y \notin U_q$ is generated as common zero locus of n independent random sections under the q^{th} -measure, is rejected from the point set. This means that there exists another subset U_r and metric ω_{FS}^r , with $y \in U_r$, such that n -tuples of random sections under the r^{th} -measure, generate points on U_r more closely distributed under $\nu \wedge \bar{\nu}$.

There is not a unique answer to the question of how to generate optimal sets of Fubini-Study metrics and subsets on X when $Q > 1$. We propose a method that is useful in numerical applications. First we need a definition.

Definition 2. Given a point $x \in X$, there exists a x -mass one Fubini-Study metric $\omega_{FS}(\lambda_x)$ on $\mathbb{P}H^0(X, \mathcal{L})^*$, that satisfies

$$n! \frac{\nu \wedge \bar{\nu}}{i^* \omega_{FS}(\lambda_x)^n}(x) = 1.$$

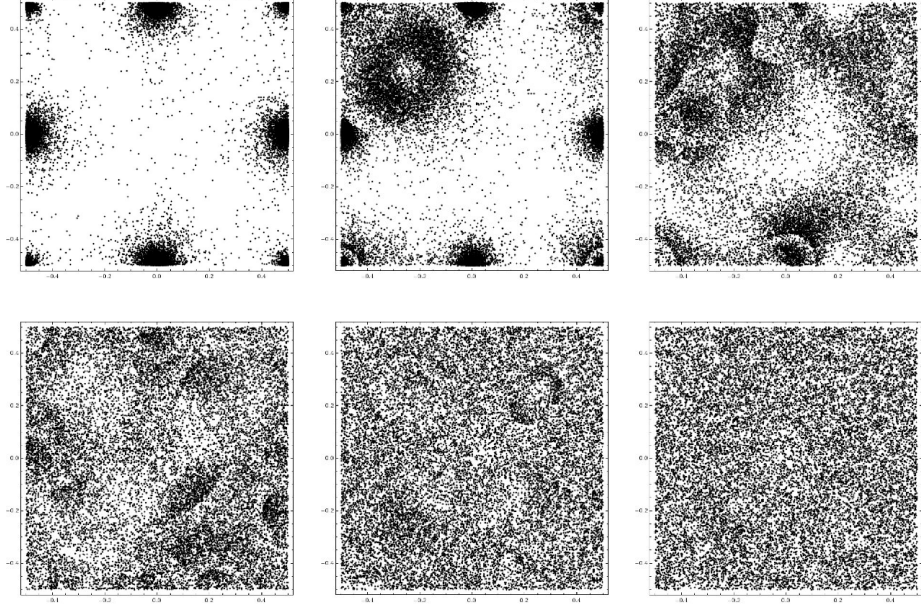


FIGURE 3. Distribution of 20,000 random points on the Weierstrass cubic, for 1, 2, 3, 5, 11, and 19 Fubini-Study metrics & subsets, optimally chosen.

The construction of $\omega_{FS}(\lambda_x)$ goes as follows. Let us consider an orthonormal basis $\{s_\alpha\}_{\alpha=1}^{N+1}$ for $H^0(X, \mathcal{L})$ with respect to the ν -balanced Fubini-Study metric. Now, in this basis, we introduce the matrix λ_x

$$\lambda_x = \frac{1}{1 + \epsilon} (\mathbf{1} + \epsilon P_x),$$

with $\mathbf{1}$ the identity matrix, $P_x = P_x^2$ the projector on the ray generated by $x \mapsto \mathbb{P}H^0(X, \mathcal{L})^*$, and

$$\epsilon = \left(n! \frac{\nu \wedge \bar{\nu}}{i^* \omega_{FS}(\mathbf{1})^n}(x) \right)^{\frac{1}{n}} - 1.$$

It is then straightforward to show that if

$$\log \left(\sum_{\alpha\beta} (\lambda_x^{-1})^{\bar{\beta}\alpha} s_\alpha \bar{s}_\beta \right)$$

is the Kähler potential for $\omega_{FS}(\lambda_x)$, then

$$n! \frac{\nu \wedge \bar{\nu}}{i^* \omega_{FS}(\lambda_x)^n}(x) = 1.$$

We can generate optimal sets of Fubini-Study metrics by combining iteratively the refined mass formula (3.13), and the *x-mass one* metrics. Given a set of Fubini-Study metrics $\{\omega_{FS}^q\}_{q=1}^Q$ with $Q > 0$, we can add two metrics to the set

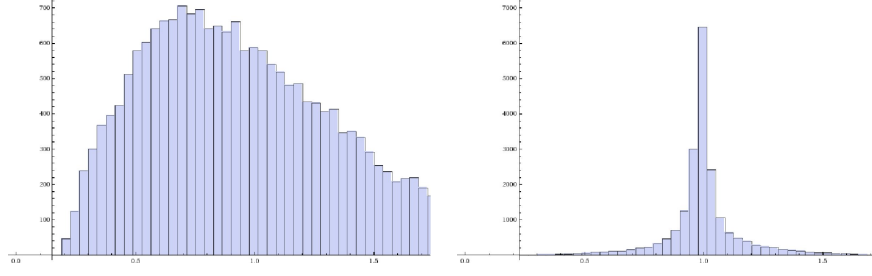


FIGURE 4. Distribution of masses for points on the Calabi-Yau Quintic $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 0.246 \times Z_0 Z_1 Z_2 Z_3 Z_4$, using 1 Fubini-Study metric (left) and 19 Fubini-Study metrics & subsets, optimally chosen (right).

by searching for the absolute maximum x_{max} and minimum x_{min} of the mass function $m_{IPS}(x, \{\omega_{FS}^q\}_{q=1}^Q)$, and adding $\omega_{FS}(\lambda_{x_{min}})$ and $\omega_{FS}(\lambda_{x_{max}})$ to the set:

$$\{\omega_{FS}^q\}_{q=1}^{Q+2} \text{ such that } \omega_{FS}^{Q+1} = \omega_{FS}(\lambda_{x_{max}}), \omega_{FS}^{Q+2} = \omega_{FS}(\lambda_{x_{min}}).$$

Fig. 3 and Fig. 4 show a few examples of Improved Point Sets on the Weierstrass cubic defined above, and on a Quintic 3-fold.

3.2. Example: the family of Quintics. Equipped with equations (3.3), (3.9) and the Monte Carlo integration technique just described, one can estimate Weil-Petersson metrics for a large class of families. For instance, one can compute the metric on the modulus of Quintic 3-folds introduced above,

$$(3.14) \quad P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4,$$

and studied independently by [COGP]. By generating 2,000,000 points we evaluated the Weil-Petersson metric at the Fermat point $t = 0$:

$$(3.15) \quad g_{t\bar{t}}(0) = 0.19205 \pm 0.00104,$$

with 0.00104 the standard error. Eq. (3.15) should be compared with the exact value (0.1922...), obtained by computing the volume of the Quintic, as function of t , via integration of its 3-cycles, (2.4). In Fig. 5 we computed the Weil-Petersson metric in the same region of the t -plane studied in Fig. 1.

Another algorithm that allows to estimate the Weil-Petersson metric consists in evaluating the logarithm of numerical volumes for several 3-folds near the manifold that we want to study, fitting a quadratic function for the values therein, and computing its Hessian. This method is highly inefficient despite is much simpler to implement. As an example, if we evaluate the function

$$(3.16) \quad F(t, \bar{t}) = -\log \left(\int_Q \nu_t \wedge \bar{\nu}_t \right),$$

for 300 random values of t near $t = 0$ with 100,000 points on each 3-fold Q_t , and fit a quadratic function around the Fermat point, we find that the Hessian at $t = 0$ is

$$g_{t\bar{t}}(0) = 0.209693 \pm 0.03.$$

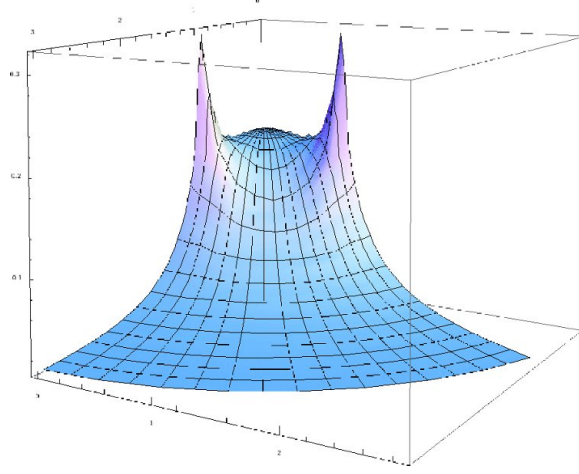


FIGURE 5. Weil-Petersson metric (vertical axis) on the t -plane (horizontal plane) of Calabi-Yau Quintic threefolds, $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4$, computed via local deformations of the holomorphic form and Monte Carlo integrals.

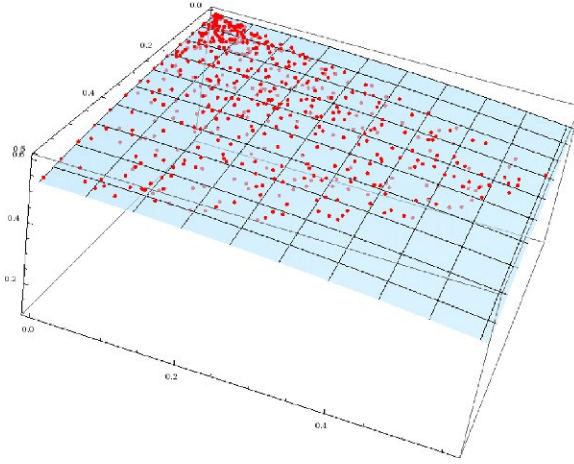


FIGURE 6. Quadratic fit of the function $-\log(\int_X \nu_t \wedge \bar{\nu}_t)$ around $t = 0$, for 300 random Calabi-Yau Quintic 3-folds on the t plane near $t = 0$.

In other words, by using 15 times more points than in (3.15), we evaluate $g_{t\bar{t}}(0)$ with an error 30 times bigger. In Fig. 6 we represent the graph of the fitted function (3.16), for 300 points on the t -plane.

4. NUMERICAL EVALUATION VIA DONALDSON'S QUANTIZATION LINK

In this section, starting from the basics of moduli spaces of polarized manifolds, we explain how one can construct a discrete set up to approximate Weil-Petersson metrics. First of all, moduli of polarized varieties can be constructed as an infinite

dimensional symplectic quotient [Don2]. Let us consider the infinite dimensional space of complex structures $\mathcal{J} = \{J: TX \rightarrow TX : J^2 = -1\}$ which make (X, ω) Kähler, and certain complexification of the group of Hamiltonian diffeomorphisms $\text{Ham}_c(X, \omega)$. If one assumes $H^1(X, \mathbb{R}) = 0$ for simplicity, $\text{Ham}_c(X, \omega)$ consists in the set of diffeomorphisms of X such that the pullback of ω is compatible with the complex structure J :

$$\{f: X \longrightarrow X : \exists Q \in C^\infty(X, \mathbb{R}) \text{ such that } f^*\omega = \omega + 2i\partial\bar{\partial}Q\}.$$

The “quotient” of \mathcal{J} by $\text{Ham}_c(X, \omega)$ is the set of isomorphism classes of integrable complex structures on (X, ω) .

Furthermore, the Kähler structure on X induces one on \mathcal{J} by integration. If X is endowed with a Ricci flat metric, such Kähler form is compatible with the Weil-Petersson metric on the space of metrics. The associated symplectic form is preserved by $\text{Ham}(X, \omega)$, hence one can ask for a moment map. Fujiki and Donaldson showed that

$$\text{Moment map} = \text{scal}(\omega) - \text{scal}_0.$$

With $\text{scal}(\omega)$ the scalar curvature and scal_0 its average over X . Thus, for manifolds X with trivial canonical bundle, the zeros of the Moment map correspond to Kähler Ricci flat metrics.

4.1. Quantization of the G.I.T problem for cscK metrics. In [Don4], Donaldson showed how this previous infinite dimensional quotient should be thought as the classical limit of a finite dimensional construction.

Let us fix a polynomial $\chi(T) \in \mathbb{Q}[T]$ of degree n . One can consider the set \mathcal{H}_χ formed by couples (V, \mathcal{L}_V) such that V is a projective variety of complex dimension n and \mathcal{L}_V a polarization with Euler-Poincaré characteristic $\chi(V, \mathcal{L}_V^k)$ satisfying

$$\chi(V, \mathcal{L}_V^k) = \chi(k)$$

for k large enough. For each element (V, \mathcal{L}_V) of \mathcal{H}_χ , one obtains an embedding (which is not unique !) in a fixed projective space

$$\iota: V \hookrightarrow \mathbb{P}H^0(V, \mathcal{L}_V^k)^* = \mathbb{P}^N$$

for k large enough, because \mathcal{H}_χ is a bounded family. Thus any projective Calabi-Yau manifold (or variety) (X, \mathcal{L}) defines a point in \mathcal{H}_χ for a suitable choice of χ . From Grothendieck’s results [Vie], there is a quasi-projective scheme $\text{Hilb}(N, \chi)$ containing \mathcal{H}_χ , the Hilbert scheme of subschemes of \mathbb{P}^N with fixed Hilbert polynomial χ . Moreover, there is a universal family $\text{Univ}_{N, \chi} = \{(V, V(x)) : x \in V\}$ over $\text{Hilb}(N, \chi)$ such that $\text{Univ}_\chi \subset \text{Hilb}(N, \chi) \times \mathbb{P}^N$, i.e. one has the diagram

$$\begin{array}{ccc} \text{Univ}_{N, \chi} & \xrightarrow{\pi_2} & \mathbb{P}^N \\ \downarrow \pi_1 & & \\ \mathcal{H}_\chi & & \end{array}$$

Let us consider \mathcal{H}_χ^∞ the subset of \mathcal{H}_χ formed by smooth projective manifolds. Hence, one can consider a natural Kähler structure on \mathcal{H}_χ^∞ by pulling-back the Fubini-Study form from \mathbb{P}^N :

$$(4.1) \quad \Omega_k = \pi_{1*}(\pi_2^* \omega_{FS}^{n+1} \cap [\text{Univ}]).$$

This corresponds to write at the point $X \in \mathcal{H}_\chi^\infty$,

$$(4.2) \quad \Omega_k(v_1, v_2) = \int_X \omega_{FS}(v_1, v_2) \frac{i^n \omega_{FS}^n}{n!},$$

with v_1, v_2 vector fields along $T^{1,0}|_X \mathbb{P}^N$, normal to the subspace defined by the infinitesimal action of $SL(N+1, \mathbb{C})$. More precisely, if $\Gamma(T^{1,0}|_X \mathbb{P}^N)$ denotes the space of smooth $(1, 0)$ vector fields on \mathbb{P}^N restricted to $X \subset \mathbb{P}^N$, then $\Gamma(T^{1,0}|_X \mathbb{P}^N)$ decomposes, under the L^2 metric inherited from ω_{FS} on \mathbb{P}^N , as a direct sum

$$\Gamma(T^{1,0}|_X \mathbb{P}^N) = \Gamma(\text{Lie}(SL(N+1, \mathbb{C}))|_X) \oplus \Gamma(\text{Lie}(SL(N+1, \mathbb{C}))|_X)^\perp.$$

Here, $\Gamma(\text{Lie}(SL(N+1, \mathbb{C}))|_X)$ denotes the standard infinitesimal action of $SL(N+1, \mathbb{C})$ on \mathbb{P}^N restricted to $X \subset \mathbb{P}^N$. Thus, $v_1, v_2 \in \Gamma(\text{Lie}(SL(N+1, \mathbb{C}))|_X)^\perp \subset \Gamma(T^{1,0}|_X \mathbb{P}^N)$ in (4.2) are normal to $\Gamma(\text{Lie}(SL(N+1, \mathbb{C}))|_X)$.

Of course, from the natural action of $SL(N+1, \mathbb{C})$ over \mathbb{P}^N , the group $SL(N+1, \mathbb{C})$ will act equivariantly on $\text{Hilb}(N, \chi)$ and $\text{Univ}_{N, \chi}$. With respect to Ω_k , this leads to a natural moment map

$$\mu: \mathcal{H}_\chi^\infty \rightarrow \text{Lie}(SU(N+1))^*.$$

Given an orthonormal basis of sections $\{s_\alpha\}$ of $H^0(X, \mathcal{L}^k)$, one can write this associated moment map as

$$(4.3) \quad \mu(X) = \delta_{\alpha\bar{\beta}} - \frac{N+1}{\text{Vol}(X)} \int_X \frac{s_\alpha \bar{s}_\beta}{\sum_\gamma |s_\gamma|^2} \frac{\omega_{FS}^n}{n!}$$

The zeros of μ correspond to “balanced” manifolds $(X, \mathcal{L}^k) \in \text{Hilb}(N, \chi)^{ps}$ that are polystable in the sense of G.I.T. For those manifolds, there exists an embedding for which the center of mass with respect to the Fubini-Study form is zero. One can reformulate this by considering two natural maps on the space of metrics over \mathcal{L}^k and the space of metrics over $H^0(X, \mathcal{L}^k)$:

- The ‘Hilbertian’ map,

$$\text{Hilb}_k: \text{Met}(\mathcal{L}^k) \rightarrow \text{Met}(H^0(X, \mathcal{L}^k))$$

such that

$$\text{Hilb}_k(h)(s, \bar{s}) = \int_X |s|_h^2 \frac{c_1(h)^n}{n!}$$

with $c_1(h)$ the curvature of h .

- The injective ‘Fubini-Study’ map,

$$FS_k: \text{Met}(H^0(X, \mathcal{L}^k)) \rightarrow \text{Met}(\mathcal{L}^k)$$

such that

$$FS_k(H) = \frac{1}{k} \log \left(\sum_{i=1}^{N_k} |S_i^H|^2 \right),$$

where S_i^H is an H -orthonormal basis¹ of holomorphic sections of $H^0(X, \mathcal{L}^k)$.

Equivalently, $\mu(X) = 0$ corresponds to the existence of a Hermitian metric H_k on the vector space $H^0(X, \mathcal{L}^k)$ such that $\text{Hilb}_k(FS_k(H_k)) = H_k$, i.e to a fixed point of the map $T = \text{Hilb}_k \circ FS_k$ where

$$T: \text{Met}(H^0(X, \mathcal{L}^k)) \rightarrow \text{Met}(H^0(X, \mathcal{L}^k)).$$

¹note that this definition is independent of the choice of the basis.

Definition 3. We say that a fixed point $H_k \in \text{Met}(H^0(X, \mathcal{L}^k))$ of the T -map is a balanced metric of order k . We will also say that the induced metrics $FS_k(H_k) \in \text{Met}(\mathcal{L}^k)$ and $i_k^* \omega_{FS} = \frac{1}{k} c_1(FS_k(H_k))$ are balanced.

In [Don4, Don5, San], it is proved that for any smooth manifold X with a cscK metric in the class $c_1(\mathcal{L})$ and with $\text{Aut}(X, \mathcal{L})$ group discrete, there exists a balanced metric $H_k \in \text{Met}(H^0(X, \mathcal{L}^k))$ for k sufficiently large. The sequence of Kähler forms $c_1(FS_k(H_k))^{1/k}$ converges, in C^∞ topology and when $k \rightarrow \infty$, to the unique cscK metric in its class. Moreover, the T map admits a unique attractive fixed point, the balanced metric H_k and the iteration of the T map converges exponentially fast to the balanced metric. In this way one can obtain a complete analog of the infinite dimensional quotient $\mathcal{J}/\text{Ham}_c(X, \omega)$ defined above, using a finite dimensional framework. The idea of approximating Kähler metrics by Fubini-Study metrics via projective *balanced* embeddings, goes back to [BLY].

From now, we will restrict our attention to the smooth orbits of the moduli space $\mathcal{M}_{N, \chi} = \text{Hilb}(N, \chi)^{ps} // SL(N+1)$. In the case of polarized Calabi-Yau manifolds, the dimension of this quotient is given by $\dim H^1(X, TX)_{\mathcal{L}}$, i.e. elements of $H^1(X, TX)_{\mathcal{L}}$ that keep the polarization \mathcal{L} invariant. Thus, $\mathcal{M}_{N, \chi}$ is strictly included in the Kuranishi space of deformations of the manifold, leaving some non-algebraic deformations. Also, we can consider \mathcal{M}_H the moduli space of isomorphism classes of polarized Hodge manifolds (V, \mathcal{L}_V) with $(V, c_1(\mathcal{L}_V))$ diffeomorphic to (V_0, α) with $\alpha \in H^2(V_0, \mathbb{R})$. Then, in the particular case of Calabi-Yau's, the moduli space \mathcal{M}_H carries a structure of orbifold complex space and $\mathcal{M}_{N, \chi} \subset \mathcal{M}_H$ is open and closed in \mathcal{M}_H [FS, Section 5 and 11].

On $\mathcal{M}_{N, \chi}$, we shall see that there exists a quantized Weil-Petersson metric obtained from (4.1) by restriction. If X carries a cscK metric in the class of \mathcal{L} , then we know that there exists a convergent sequence of balanced metrics in $c_1(\mathcal{L})$, where each element of the sequence corresponds to a point in $\mathcal{M}_{N, \chi}$. For the purpose of this paper, we consider local algebraic deformations of the complex structure on X , which correspond to tangent vectors in $\mathcal{M}_{N, \chi}$. An integrable complex structure $J \in \mathcal{J}_{int}$ on X can be deformed by any element $v \in \Omega^{0,1} T^{(0,1)} X_J$. We can be more precise by fixing the differentiable structure on X and the complex line bundle L^k , and considering integrable complex structures on X with corresponding Dolbeault operators on L^k . If L^k is a fixed complex line bundle on X and J is a complex structure on X , we say that \mathcal{L}^k is the associated holomorphic line bundle on X_J endowed with a Dolbeault operator $\bar{\partial} = \bar{\partial}_J$. The deformation of the complex structure $J + v$ on X induces a deformation of the Dolbeault operator $\bar{\partial} = \bar{\partial}_J: \Omega^{p,q}(\mathcal{L}^k) \rightarrow \Omega^{p,q+1}(\mathcal{L}^k)$ as

$$\bar{\partial}_{J+v} = \bar{\partial} + v\partial + O(v^2).$$

With h a Hermitian metric on L , we obtain a L^2 metric on $\Omega^0(X, L^k)$; we denote by $L^2(X, L^k)$ its L^2 completion. If the dimension $\dim \ker \bar{\partial}_J = N+1$ is constant as J varies on \mathcal{J}_{int} , there exists a natural embedding of \mathcal{J}_{int} :

$$(4.4) \quad \tau: \mathcal{J}_{int} \rightarrow \text{Gr}(N+1, L^2(X, L^k)),$$

with $\text{Gr}(N+1, L^2(X, L^k))$ the Grassmannian of $N+1$ planes in $L^2(X, L^k)$, and $J \mapsto \ker \bar{\partial}_J \subset L^2(X, L^k)$ for every element $J \in \mathcal{J}_{int}$. If (s_α) is an orthonormal basis of sections for the finite dimensional vector space $\ker \bar{\partial}_J$, the infinitesimal deformation v is pushforwarded to a tangent vector on the Grassmannian $\text{Gr}(N+1, L^2(X, L^k))$

under (4.4). The induced vector can be computed as the infinitesimal deformation of the basis of sections (s_α) . Thus, if $(s_\alpha + \delta s_\alpha)$ is a basis for $\ker \bar{\partial}_{J+v}$, then

$$(\bar{\partial}_J + v\partial + O(v^2))(s_\alpha + \delta s_\alpha) = 0$$

and $\delta s_\alpha = -\bar{\partial}_J^{-1}(v\partial s_\alpha)$, neglecting $O(v^2)$ corrections. One has to define the inverse $\bar{\partial}_J^{-1}$ properly on the orthogonal complement $\ker \bar{\partial}_J^\perp \subset L^2(X, L^k)$. As δs_α also denotes a tangent vector to $\text{Gr}(N+1, L^2(X, L^k))$ in homogeneous coordinates, the $(N+1)$ -plane spanned by the $\bar{\partial}_J^{-1}(v\partial s_\alpha)$ in $L^2(X, L^k)$, is orthogonal to $\ker \bar{\partial}_J$. Therefore, one can alternatively define (4.2) as follows.

Definition 4. For $H_t \in \text{Met}(H^0(X_t, \mathcal{L}^k))$ the balanced metric of order k , and for $v_1, v_2 \in \Omega^{0,1}T^{(0,1)}X$ representatives of $v_i \in T_{J_0}\mathcal{J}_{int} \simeq H_{\bar{\partial}}^{0,1}(TX_t)$, we define the quantized Kähler form Ω_k in $\Omega^{1,1}(T)$ as,

$$(4.5) \quad \Omega_k(v_1, v_2) = k^n H_t^{\alpha, \beta} \int_{X_t} \langle \bar{\partial}^{-1}(v_1 \partial s_\alpha), \bar{\partial}^{-1}(v_2 \partial s_\beta) \rangle_{FS(H_t)} \frac{(\frac{1}{k} c_1(FS(H_t)))^n}{n!}$$

for (s_α) an orthonormal basis with respect to H_t .

Theorem 2. *As k tends to infinity, the sequence of Kähler forms Ω_k converges to the Weil-Petersson metric defined in Equation (2.1), in the C^∞ topology, over the smooth points of the moduli space $\mathcal{M}_{N, \chi}$.*

Proof. Let us introduce \mathbf{G} the Green operator, inverse of the Kodaira Laplacian Δ restricted to the space $(0, 1)$ -forms. Let us denote $G_k(x, y)$ its integral kernel with respect to the volume form $\omega_{FS(H)}^n = (\frac{1}{k} c_1(FS(H)))^n / n!$ induced by the balanced metric H . Then, since $\bar{\partial} + v\partial$ is a new integrable complex structure, one obtains

$$(\bar{\partial} + v\partial)^2 = \bar{\partial}(v\partial) + (v\partial)\bar{\partial} + (v\partial)^2 = 0.$$

Thus, we obtain that for any holomorphic section s , at first order in v ,

$$\Delta(v\partial)s = (\bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*)(v\partial)s = \bar{\partial} \bar{\partial}^*(v\partial)s + O(v^2).$$

Finally, by the definitions of \mathbf{G} and Ω_k , (4.5), we get

$$\begin{aligned} \Omega_k(v_1, v_2) &= k^n H_t^{\alpha, \beta} \int_X \langle \mathbf{G}(v_1 \partial) s_\alpha(x), v_2 \partial s_\beta \rangle_{FS(H_t)} \omega_{FS(H_t)}^n, \\ &= k^n H_t^{\alpha, \beta} \int_X \int_X \langle G_k(x, y) v_1 \partial s_\alpha(x), v_2 \partial s_\beta(y) \rangle_{FS(H_t)} \omega_{FS(H_t)}^n(x) \omega_{FS(H_t)}^n(y). \end{aligned}$$

Hence, we are lead to give an asymptotic of the Green kernel when $k \rightarrow +\infty$. We can prove that the sequence of Kähler metrics Ω_k converges to the Weil-Petersson metric, if we can show that $G_k(x, y)$ converges, up to rescaling, to $\frac{1}{k^n} \delta(x, y) + O(\frac{1}{k^{n+1}})$, with $\delta(x, y)$ the Dirac function on the diagonal. This is precisely the technical statement that we obtain next section (Theorem 3). As we shall see, the proof consists essentially in a localization argument. \square

4.1.1. Asymptotics of the Green kernel: Kodaira Laplacian on $(0, 0)$ -forms. For now, we are interested in the Green operator \mathbf{G} associated to the Kodaira Laplacian $\Delta = (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial})$ acting on sections with values in \mathcal{L}^k , i.e $C^\infty(X, \mathcal{L}^k)$. In next subsection, we will deal with the general case of $(0, q)$ -forms. We denote by \mathbf{G}_q the associated Green operator restricted to $(0, q)$ -forms; although, in order to prove Theorem 2, we just need to consider $(0, 1)$ forms and the asymptotics of \mathbf{G}_1 . Formally, \mathbf{G} can be defined as the inverse of the Kodaira Laplacian $\Delta \mathbf{G} = Id$. We

want to describe the asymptotics of its integral kernel $G_k(x, y)$ in the limit $k \gg 1$; especially the term along the diagonal i.e $G_k(x, x)$ for $x \in X$. We express the action of \mathbf{G} on an element $f \in C^\infty(X, \mathcal{L}^k)$ as

$$\mathbf{G}(f)(x) = \int_M G_k(x, y) f(y) dV_y.$$

We describe the asymptotics of $G_k(x, y)$ using a 3 step argument:

(1) **Reducing the problem.** First of all we consider the case of the Green kernel for sections of \mathcal{L}^k . Let us denote λ_j the eigenvalues of Δ . Then, there is a spectral gap for the Kodaira Laplacian operator when $k \rightarrow \infty$, which means that $\lambda_j \in \{0\} \cup [C_0 k + C_1, \infty[$. This is essentially a consequence of the Bochner-Kodaira-Nakano formula and Hörmander's $\bar{\partial}$ -theory and is the crucial argument to localize our computation. We refer to [M-M, Sections 1.4 and 1.5] as a survey on this topic. Let us denote π_{λ_j} the L^2 projection with respect to h and the volume form $\frac{\omega^n}{n!}$ from the space $C^\infty(X, \mathcal{L}^k)$ to the eigenspace of Δ associated to the eigenvalue λ_j . Then, if we set $\lambda_0 = 0$,

$$G_k(x, x) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \pi_{\lambda_j} \in \text{End}(\mathcal{L}^k),$$

one can write

$$G_k(x, y) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \sum_{i=1}^{N_j} S_i^{(j)}(x) \otimes (S_i^{(j)}(y))^*,$$

where $(S_i^{(j)})$ form an L^2 -orthonormal basis of $\text{Ker}(\Delta - \lambda_j \text{Id})$.

(2) **The computation is local.** One needs to see that if $G_k^0(x, x)$ is the Green kernel obtained by freezing the coefficients of the involved operators and metric, for $x = x_0$, then $G_k(x, x)$ is asymptotically close to $G_k^0(x, x)$ at $x = x_0$. One knows that

$$\Delta_k^0 - \Delta_k = O\left(\frac{1}{k}|x - x_0|^2 + \left(\frac{1}{k} + |x - x_0|^2\right)\nabla + |x - x_0| + \frac{1}{k}\right),$$

and therefore

$$|G_k^0(x_0, x_0) - G_k(x_0, x_0)| = \left(\sum_{j>1} \frac{|\lambda_j^0 - \lambda_j|}{\lambda_j^0 \lambda_j}\right) \mathbf{O}(1/k).$$

By invoking the spectral gap theorem, we have shown that the calculation is local.

(3) **The local computation.** In this step we obtain the asymptotics of the local operator for $k \rightarrow \infty$. First we express G^0 using Fourier transforms. Let us consider the local Laplacian on the manifold seen as a real manifold.

$$\Delta = -\frac{\partial^2}{dx^2} + a^2 x^2.$$

Note that a will be fixed later given by the condition $k \times \gamma > 0$ where $\omega = \sqrt{-1} \sum_{i=1}^n \gamma_i dz_i \wedge d\bar{z}_i$. Now, the eigenfunctions of Δ associated to the eigenvalues $\lambda_j = (2j+1)a$ are given by

$$\left(\frac{\sqrt{a}}{\sqrt{\pi} 2^j j!}\right)^{1/2} H_j(\sqrt{ax}),$$

where H_j is the j -th Hermite polynomial given by $e^{x^2/2} \frac{\partial^j}{\partial x^j} (e^{-x^2})$. Finally, for the local operator, taking into account the fibrewise metric on \mathcal{L}^k ,

$$(4.6) \quad G_k^0(x, y) = \sqrt{\frac{a}{\pi}} e^{-\frac{a}{2}(x^2+y^2)} \sum_{j=0}^{\infty} \frac{1}{(2j+1)a^{j+1}2^j j!} \frac{\partial^j(e^{-ax^2})}{\partial x^j} \frac{\partial^j(e^{-ay^2})}{\partial y^j}.$$

We want to compute the series in the RHS of the former expression. If we denote:

$$\Sigma(x, y) = \sum_{j=0}^{\infty} \frac{1}{(2j+1)(2a)^j j!} \frac{\partial^j(e^{-ax^2})}{\partial x^j} \frac{\partial^j(e^{-ay^2})}{\partial y^j},$$

then its Fourier transform is given by

$$\mathcal{F}(\Sigma)(\eta, \zeta) = \frac{1}{2\sqrt{a}} e^{-(\zeta^2+\eta^2)/4a} \frac{\sqrt{2\pi}}{2} \operatorname{erf}\left(\sqrt{\frac{\zeta\eta}{2a}}\right) \sqrt{\frac{1}{\zeta\eta}},$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!(2i+1)}$ is the error function. We are lead to compute the inverse Fourier transform of the former expression to get G^0 , and thus the integral

$$4 \int_0^{\infty} \int_0^{\infty} \operatorname{erf}\left(\sqrt{\frac{\zeta\eta}{2a}}\right) \sqrt{\frac{1}{\zeta\eta}} e^{-(\zeta^2+\eta^2)/4a} e^{\sqrt{-1}x\zeta + \sqrt{-1}y\eta} d\eta d\zeta$$

where x, y are fixed. Let us do a change of variable:

$$\zeta = \sqrt{a}u^2, \quad \eta = \sqrt{a}v^2,$$

and the last integral is then reduced to $16\sqrt{a}I_1(x, y)$ with

$$I_1(x, y) = \int_0^{\infty} \int_0^{\infty} \operatorname{erf}\left(\frac{uv}{\sqrt{2}}\right) e^{-(u^4+v^4)/4} e^{\sqrt{-1}x\sqrt{a}u^2 + \sqrt{-1}y\sqrt{a}v^2} dudv$$

Note that it is obvious that $I_1 = O(1)$ when $a \rightarrow \infty$ and one can get $I_1(0, 0)$ by a lengthy computation. Thus we obtain the local asymptotic

$$G_k^0(x, x) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{a}} + o\left(\frac{1}{\sqrt{a}}\right).$$

Moreover, using (4.6), it is obvious that there is convergence of $G(x, y)$ for $(x, y) \neq (0, 0)$ to 0 exponentially fast.

4.1.2. Asymptotics of the Green kernel: general case and $(0, q)$ -forms.

(1) **The case of tensorizing by an arbitrary vector bundle.** Let F be an arbitrary holomorphic hermitian bundle on X . We are now extending the previous computation to $F \otimes \mathcal{L}^k$. The asymptotic of the Green kernel for $(0, q)$ -sections will be a direct consequence of our work for $F = \Lambda^{0,q} T^* M$. We consider the second order elliptic operator

$$\square = \Delta - V$$

where V is an arbitrary Hermitian endomorphism of F (that we extend trivially to $F \otimes \mathcal{L}^k$ by identifying V with $V \otimes Id_{\mathcal{L}^k}$) and Δ is the Laplacian operator acting on $F \otimes \mathcal{L}^k$. Locally, we can assume that we have an orthonormal frame in which V is diagonal and $\langle Vu, u \rangle = \sum_{i=1}^{r_F} V_i |u_i|^2$ where r_F is the rank of F . Hence, the local operator Δ of previous section is replaced by $\square = \Delta - \sum_{i=1}^{r_F} V_i Id$. We are lead to compute the Green kernel of the Green operator acting on a single component u_i

for $i = 1, \dots, r_F$. From the previous section (one can apply again the same method since there is also a spectral gap in that case [M-M, Theorem 1.5.5]), we obtain

$$G_{k,i}^0(x, y) = \sqrt{\frac{a}{\pi}} e^{-\frac{a}{2}(x^2+y^2)} \sum_{j=0}^{\infty} \frac{1}{((2j+1)a + V_i)(2a)^j j!} \frac{\partial^j(e^{-ax^2})}{dx^j} \frac{\partial^j(e^{-ay^2})}{dy^j}.$$

Let us denote $r_i = V_i/a$. Then, if we set

$$\Sigma_V(x, y) = \sum_{j=0}^{\infty} \frac{1}{(2j+1+r_i)(2a)^j j!} \frac{\partial^j(e^{-ax^2})}{dx^j} \frac{\partial^j(e^{-ay^2})}{dy^j},$$

its Fourier transform is:

$$\mathcal{F}(\Sigma_V)(\eta, \zeta) = \frac{1}{2a} e^{-(\zeta^2+\eta^2)/4a} 2^{\frac{r_i-1}{2}} \frac{1}{(\zeta\eta)^{\frac{r_i+1}{2}}} a^{\frac{r_i+1}{2}} \int_0^{\frac{\zeta\eta}{2a}} e^{-t} t^{\frac{r_i-1}{2}} dt.$$

By the same change of variables,

$$\zeta = \sqrt{a}u^2, \quad \eta = \sqrt{a}v^2,$$

we obtain,

$$\begin{aligned} G_{k,i}^0(x, y) &= e^{-\frac{a(x^2+y^2)}{2}} \frac{1}{\sqrt{\pi}} \frac{2^{\frac{r_i+3}{2}} a^{r_i/2+1}}{2a \times a^{\frac{r_i+1}{2}}} \times \\ &\quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(uv)^{r_i}} e^{\sqrt{-1}\sqrt{a}xu^2} e^{\sqrt{-1}\sqrt{a}yv^2 - \frac{u^4+v^4}{4}} \int_0^{\frac{u^2v^2}{2}} e^{-t} t^{\frac{r_i-1}{2}} dt du dv. \end{aligned}$$

Which finally is

$$\begin{aligned} G_{k,i}^0(x, y) &= e^{-\frac{a(x^2+y^2)}{2}} \sqrt{\frac{2}{a\pi}} \times \\ &\quad 2^{r_i} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{\sqrt{-1}\sqrt{a}(xu^2+yv^2) - \frac{u^4+v^4}{4}}}{(uv)^{r_i}} \int_0^{\frac{u^2v^2}{2}} e^{-t} t^{\frac{r_i-1}{2}} dt du dv. \end{aligned}$$

Note that $r_i \rightarrow 0$ when $k \rightarrow +\infty$. Again, if $(x, y) \neq (0, 0)$, the term $G_{k,i}(x, y)$ tends to 0 exponentially fast. The Green kernel associated to the operators $\square|_{Vect(u_i)}$ that are pairwise commuting since they act on disjoint sets of variables, is given by the product of Green kernels $G_{k,i}$.

(2) **Application to the Green kernel for $(0, q)$ -forms.** Consider that the curvature of the Hermitian metric on \mathcal{L} is given locally by $\omega = \sqrt{-1} \sum_{j=1}^n \gamma_j dz_j \wedge d\bar{z}_j$. Let V be the endomorphism of $F \otimes \mathcal{L}^k$ with $F = \Lambda^{(0,q)} T_X^*$, which is diagonal in the basis $(d\bar{z}_J)$ where $|J| = q$ and with eigenvalues:

$$\sum_{i \notin J} \gamma_j - \sum_{i \in J} \gamma_j.$$

We apply the result of the previous section, and we obtain the local Green kernel (restricted to the diagonal) associated to the Laplacian acting on $(0, q)$ -forms with values in \mathcal{L}^k over X with $\dim_{\mathbb{R}}(X) = 2n$. Our results are summed up with the following theorem.

Theorem 3. *The Green kernel $G_{k,q}$ associated to the Green operator \mathbf{G}_q acting on $(0, q)$ -forms of L^k has an asymptotic expansion when $k \rightarrow +\infty$,*

$$G_{k,q}(x, x) \sim \frac{1}{k^n} \frac{2^n}{\pi^n} \sum_{|J|=q} \prod_{i=1}^n \frac{2^{r_{J,i}}}{\gamma_i(x)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^4+v^4}{4}}}{(uv)^{r_{J,i}(x)}} \int_0^{\frac{u^2v^2}{2}} t^{\frac{r_{J,i}(x)-1}{2}} \frac{1}{e^t} dt du dv$$

where one has set $\omega = c_1(h) = \sqrt{-1} \sum_{j=1}^n \gamma_j(x) dz_j \wedge d\bar{z}_j$ at the point x and

$$r_{J,i}(x) = \frac{\sum_{i \notin J} \gamma_j(x) - \sum_{i \in J} \gamma_j(x)}{k \gamma_i(x)}.$$

The convergence is uniform in the variable x for the C^∞ topology and for the metric h on L varying in a compact set (in C^∞ topology).

Moreover $G_{k,q}(x, y)$ is converging exponentially fast to 0 for $x \neq y$ when $k \rightarrow +\infty$.

For numerical applications, we will use the fact that

$$G_{k,q}(x, y) \simeq \frac{4}{k^n} \left(\frac{2}{\pi}\right)^n \binom{n}{q} \frac{1}{\prod_{i=1}^n \gamma_i(x)} \delta(x, y) + O\left(\frac{1}{k^n}\right).$$

4.1.3. Quantized Weil-Petersson metric for projective Calabi-Yau manifolds. If we consider a holomorphic family of projective Calabi-Yau manifolds, we can modify the notion of quantized Weil-Petersson metric of Definition 4, by substituting the Fubini-Study volume form by $\nu \wedge \bar{\nu}$. This modification has important advantages in numerical applications. First, we review the ν -balanced metrics introduced in [Don6].

Definition 5. Let (X, \mathcal{L}) be a polarized Calabi-Yau manifold and ν a $(n, 0)$ -form. For $k \gg 0$, let us consider the map $Hilb_{k,\nu}: \text{Met}(\mathcal{L}^k) \rightarrow \text{Met}(H^0(X, \mathcal{L}^k))$ given by

$$Hilb_{k,\nu}(h) = \int_X h(\cdot, \cdot) \nu \wedge \bar{\nu}.$$

We say that a fixed point H_k of the map $T_\nu = Hilb_{k,\nu} \circ FS_k$ is a ν -balanced metric of order k .

The advantage of working with ν -balanced metrics is that, contrarily to the cscK case, the map $T_\nu = Hilb_{k,\nu} \circ FS_k$ admits always a unique fixed attractive point which ensures the existence and uniqueness of the ν -balanced metric of order k .

Definition 6. With the notations of Section 2.1, let $H_t \in \text{Met}(H^0(X_t, \mathcal{L}^k))$ be the ν_t -balanced metric of order k for k sufficiently large. For $v_1, v_2 \in \Omega^{0,1} T^{(0,1)} X_t$, one has a Kähler form on \mathcal{T} defined as

$$(4.7) \quad \Omega_k(v_1, v_2) = k^n H_t^{\alpha,\beta} \int_{X_t} \langle \bar{\partial}^{-1}(v_1 \partial s_\alpha), \bar{\partial}^{-1}(v_2 \partial s_\beta) \rangle_{FS(H_t)} \nu_t \wedge \bar{\nu}_t$$

for (s_α) an orthonormal basis with respect to H_t . Making an abuse of notation, we also call Ω_k the quantized Weil-Petersson metric of order k , as in Eq. (4.2).

In order to see that Ω_k converges to the Weil-Petersson metric, we just need to check that the volume form $\omega_{FS(H_t)}^n$ converges to $\nu_t \wedge \bar{\nu}_t$, and to invoke Theorem 2. As $\omega_{\infty,t}^n = \nu_t \wedge \bar{\nu}_t$, with $\omega_{\infty,t} \in c_1(\mathcal{L})$, is the Calabi-Yau metric, we know from

the construction of the balanced metrics (more precisely from the asymptotic of the Bergman kernel [M-M, Remark 5.1.5]) that

$$\|\omega_{FS(H_t)} - \omega_{\infty,t}\|_{C^\infty} = O\left(\frac{1}{k^2}\right).$$

In particular,

$$\frac{\omega_{FS(H_t)}^n}{\nu_t \wedge \bar{\nu}_t} = 1 + O\left(\frac{1}{k^2}\right),$$

and therefore, Ω_k converges to the Weil-Petersson metric of Definition 1.

Theorem 4. *Given a family of polarized Calabi-Yau varieties, the sequence of Kähler forms Ω_k defined in Def. 6 converges, in the limit $k \rightarrow +\infty$ and under the C^∞ topology, to the Weil-Petersson metric over the smooth points of the moduli space (as defined in Def. 1).*

4.2. Generalizations to other moduli spaces. We have defined quantized Weil-Petersson metrics for moduli of cscK manifolds and/or Calabi-Yau manifolds. In this section, we explain how one can define quantized Weil-Petersson metrics on other moduli spaces. We believe that, in some situations, these ideas should be useful for numerical applications. One possible application of these techniques could emerge in flux compactifications of string theory. For instance, recently J-X. Fu and S.T. Yau have studied flux compactifications of string theories that lead to complex Monge-Ampère equations over non projective manifolds. Furthermore, the approach we introduce below should enable us to get some information of the geometric structure of the moduli space, by extending Weil-Petersson metrics as currents on the semi-stable locus.

4.2.1. The non projective case. In this section, we briefly explain how to construct balanced metrics for Calabi-Yau manifolds which are not necessarily projective. We will deduce from Section 4.1 a way to define Weil-Petersson metrics for general Calabi-Yau manifolds. Although Ricci flat metrics transform trivially along rays of the Kähler cone and one can always approximate real rays of the Kähler cone by rational rays (and hence, approximate non-projective manifolds by projective ones), it is interesting to develop techniques that allow to study Calabi-Yau manifolds which are intrinsically non-projective.

Let us consider ν a $(n,0)$ -form on a Calabi-Yau manifold X . In [Kel], it is proved that the sequence of ν -balanced metrics in an integral class converges to the solution of the Calabi problem, that is the Kähler metric ω_∞ solution of the complex Monge-Ampère equation

$$(4.8) \quad \omega_\infty^n = \nu \wedge \bar{\nu}.$$

We explain now how one can build a sequence of balanced metrics if there is no polarization on the manifold. Given a Calabi-Yau manifold X and a Kähler class $\alpha > 0$, one can consider a $Spin^c$ structure \mathcal{L}_c with $c_1(\mathcal{L}_c) = -K_X + \alpha = \alpha$ [Hit]. Using the Clifford connexion on $\Lambda(T^{*(0,1)}X)$, it is possible to construct a $Spin^c$ Dirac operator

$$D^c: \Omega^{0,\cdot}(X, \mathcal{L}_c) \rightarrow \Omega^{0,\cdot}(X, \mathcal{L}_c)$$

which is a first order differential operator. Now, the fact that α is positive induces a spectral gap for D^c when considering higher tensor powers of \mathcal{L}_c . This means,

that for the induced Dirac operator D_k^c on $\Omega^{0,\cdot}(X, \mathcal{L}_c^k)$

$$\text{Spec}((D_k^c)^2) \subset \{0\} \cup [kC_0 + C_1, +\infty[$$

for two fixed constants C_0, C_1 and k sufficiently large [M-M, Theorem 1.4.8]. Furthermore, one can consider for k sufficiently large, the Kernel $\ker(D_k^c)^2 = \ker D_k^c$ which is a finite dimensional vector space (since X is compact). The Bergman kernel is the kernel of the orthogonal projection from $L^2(X, \mathcal{L}_c)$ to $\ker D_k^c$ with respect to L^2 metric induced by the fibrewise metric h_L and the volume form $\nu \wedge \bar{\nu}$. One can write it as

$$B_k(h_L)(x, y) = \sum_{i=1}^{\dim \ker D_k^c} S_i(x) \otimes S_i(y)^{*_{h_L}} \in C^\infty(X \times X, \Lambda T^{*(0,1)} X)$$

where (S_i) forms an orthonormal basis of $\ker D_k^c$.

Definition 7. Given (X, ω, J) a compact Kähler manifold with complex structure J and Kähler form ω , and given ν a $(n, 0)$ form, we define a ν -balanced metric $h_k \in \text{Met}(\mathcal{L}_c^k)$ of order k , a Hermitian metric on the Spin^c structure \mathcal{L}_c^k such that the associated Bergman distortion function is constant

$$B_k(h_k)(x, x) = \frac{\dim \ker D_k^c}{\int_X \nu \wedge \bar{\nu}}$$

for all $x \in X$.

From the asymptotic expansion given in [M-M, Theorem 4.1.3], it is clear that if the normalized sequence of ν -balanced metrics $(h_k)^{1/k}$ converges in smooth topology then the limit has flat scalar curvature, and by uniqueness, is the Calabi-Yau metric on X living in the class α .

On another hand, given a balanced metric on \mathcal{L}_c^k , one can obtain a metric on $\ker D_k^c$ by the Hilbertian map described in Section 4.1. Since the Fubini-Study map $FS_k: \text{Met}(\mathcal{L}_c^k) \rightarrow \text{Met}(\ker D_k^c)$ can also be defined, it makes sense to wonder if the induced map $T = \text{Hilb}_{k,\nu} \circ FS_k$ admits a fixed attractive point like in the projective setting. This comes directly from [Don6, Proposition 3, Proposition 4] (where one replaces the space of holomorphic sections by $\ker D_k^c$) which do not require to work with integral classes. Furthermore, we can prove that ν -balanced metrics do always exist in that context.

Theorem 5. *Given X a Calabi-Yau manifold, ν a $(n, 0)$ -form and α a Kähler class on X , there exists for k sufficiently large a ν -balanced metric h_k of order k . When k tends to infinity the curvature of $(h_k)^{1/k}$ converges in smooth topology to the Calabi-Yau metric on X in the class α .*

Proof. The proof is similar to [Kel, Theorem 3] and we just focus on the main differences with it. One needs to extend the work of X. Wang [Wan2, Theorem 1.2] to our situation and apply it with the trivial line bundle. From Yau's solution to the Calabi conjecture [Yau], we know that in the Kähler setting, we can solve Equation (4.8), and this is the crucial argument of the proof. Then, one notices that the ν -balanced metric is a balanced metric in the sense of Wang for the Kähler metric ω_∞ . Starting with \mathcal{E} the trivial line bundle over X , one can modify like in the proof of [Wan2, Theorem 4.1] the metric on \mathcal{L}_c^k in order to force the distortion

function to satisfy:

$$B_k(h_k)(x, x) = \frac{\dim \ker D_k^c}{\int_X \nu \wedge \bar{\nu}} \left(1 + O\left(\frac{1}{k^q}\right) \right).$$

for a fixed integer $q > 0$. The key ingredient here is that one can deform inductively the trivial metric on \mathcal{E} and that this deformation is controled by the operator $\sqrt{-1}\Lambda\partial\bar{\partial}$ which is invertible [LT]. This gives a sequence of “approximate” ν -balanced metric on $\text{Met}(\mathcal{E} \otimes \mathcal{L}_c^k) = \text{Met}(\mathcal{L}_c^k)$.

The second step of the proof requires to launch the gradient flow of the norm of the moment map associated to the $SU(\ker D_k^c)$ action in order to move on an $SL(\ker D_k^c)$ -orbit from the approximate ν -balanced metric to a ν -balanced metric. This is possible if one controls precisely the differential of the moment map at starting point of the flow, that it the approximate ν -balanced metric. This estimate is based actually on some L^2 -estimates involving the fact that $(D_k^c)^2$ is an elliptic operator. For instance, a consequence of the spectral gap and the localisation techniques is that for $s \in \ker(D_k^c)$, $\|\nabla^j s\|_{L^2}^2 \leq ck^{n+j}\|s\|_{L^2}^2$ for $c > 0$ a fixed constant [M-M, Sections 4.1.4, 4.1.5]. This gives in particular the analog of [Wan2, Lemma 3.1] in our setting. Moreover, using the fact that we are working with a trivial line bundle, we can use directly the basic elliptic estimate for $(D_k^c)^2$ [M-M, Theorem A.3.2] in order to obtain the analog in our situation to [Wan2, Proposition 3.4 (ii)]. Thus, we obtain all the required estimates to apply Donaldson’s argument [Don4, Proposition 17] and complete the proof of Theorem 5. \square

Then, for a holomorphic family X_t of Calabi-Yau manifolds (non necessarily projective) it is natural to define a quantized Weil-Petersson metric:

Definition 8. For $k \gg 0$ sufficiently large, let $H_t \in \text{Met}(H^0(X_t, \mathcal{L}^k))$ be the ν_t -balanced metric of order k in the sense of Definition 7. For $v_1, v_2 \in \Omega^{0,1}T^{(0,1)}X_t$, we define the following Kähler form over $T\mathcal{T}$,

$$\Omega_k(v_1, v_2) = k^n H_t^{\alpha, \beta} \int_{X_t} \langle \bar{\partial}^{-1}(v_1 \partial s_\alpha), \bar{\partial}^{-1}(v_2 \partial s_\beta) \rangle_{FS(H_t)} \nu_t \wedge \bar{\nu}_t,$$

for (s_α) an orthonormal basis with respect to H_t .

From Theorem 3, we obtain again the convergence of the sequence of Ω_k to the Weil-Petersson in C^∞ -topology.

4.2.2. The vector bundle case. The construction in Section 4.1 admits also an analog for holomorphic vector bundles equipped with Hermitian-Einstein metrics. We consider now a smooth projective manifold X of complex dimension n with polarization $(\mathcal{L}, h_{\mathcal{L}})$, and a fixed Kähler form ω_X on X . We assume for simplicity that the Picard group of X is \mathbb{Z} . Let \mathcal{E} be a holomorphic vector bundle or more generally a torsion free sheaf over X . \mathcal{E} is said to be Gieseker stable (resp. Gieseker semistable) if for any coherent subsheaf \mathcal{F} , one has the following inequalities for the respective slopes:

$$(4.9) \quad \frac{h^0(\mathcal{F} \otimes \mathcal{L}^k)}{\text{rank}(\mathcal{F})} < \frac{h^0(\mathcal{E} \otimes \mathcal{L}^k)}{\text{rank}(\mathcal{E})} \quad (\text{resp. } \leq)$$

for any k sufficiently large. A polystable bundle is just a direct sum of stable bundles.

As it is well-known, Mumford-Takemoto stable bundles are Gieseker stable, while

thanks to the celebrated Kobayashi-Hitchin correspondence, irreducible holomorphic vector bundles carry a Hermitian-Einstein metric if and only if they are Mumford-Takemoto stable. The interest of working with Gieseker stable bundles rather than Mumford stable ones is to obtain quasi-projective schemes when considering their moduli. Let us be more explicit. We fix $r = \text{rank}(\mathcal{E})$, $\det(\mathcal{E}) = \mathcal{D}$ and $h^0(\mathcal{E} \otimes \mathcal{L}^k) = \chi(k)$. We consider the space

$$\Xi = \mathbb{P}Hom\left(\bigwedge^r \mathbb{C}^{\chi(k)}, H^0(\mathcal{D} \otimes \mathcal{L}^k)\right)^*$$

equipped with the polarization $\mathcal{O}_\Xi(1)$. Then a result of D. Gieseker [Gie] asserts that for k sufficiently large, G.I.T stable (resp. semistable) points of Ξ with respect to the polarization $\mathcal{O}_\chi(1)$ and the action of $SL(\chi(k))$ correspond to Gieseker stable (resp. semistable) bundles \mathcal{E} . If the orbit is closed under the action of $SL(\chi(k))$ then the corresponding bundle is actually polystable. One can even replace Gieseker's construction in the context of Quot scheme. Let us fix a polynomial $\chi(T) \in \mathbb{Q}[T]$ of degree n and let us consider \mathcal{V} a complex vector space of finite dimension $\chi(k)$. There exists a Quot scheme

$$Quot = Quot(\mathcal{V} \otimes L^{-k}, \chi(k))$$

of torsion free quotients of the form

$$[\mathcal{V} \otimes \mathcal{L}^{-k} \rightarrow \mathcal{E} \rightarrow 0]$$

with Hilbert polynomial equal to χ , that is $h^0(\mathcal{E} \otimes \mathcal{L}^k) = \chi(k)$. If one considers $\mathfrak{S} \subset Quot$ the subscheme of all quotients which are semistable torsion free sheaves, then there is a morphism

$$\rho: \overline{\mathfrak{S}} \rightarrow \Xi$$

where $\overline{\mathfrak{S}}$ denotes the closure of \mathfrak{S} in the Quot scheme. This morphism restricted to \mathfrak{S} is injective and maps \mathfrak{S} to the set of G.I.T semistable points $\Xi^{ss} \subset \Xi$ with respect to $\mathcal{O}_\Xi(1)$ and the $SL(\chi(k))$ action [HL]. For G.I.T stable points Ξ^s , one can identify $\rho^{-1}(\Xi^s)$ with the subscheme of stable sheaves \mathfrak{S}^s inducing a quasi-projective scheme $\mathcal{M}_{\mathcal{V}, \mathcal{L}, \chi} := \mathfrak{S}^s // SL(\chi(k))$, the moduli space $\mathcal{M}_{\mathcal{V}, \mathcal{L}, \chi}$ of Gieseker stable torsion free sheaves with fixed Hilbert polynomial. Hence, our situation is summed up by the following diagram:

$$\begin{array}{ccc} Univ_{Quot} & & \\ \downarrow & & \\ \overline{\mathfrak{S}} \times X & \xrightarrow{\rho} & \Xi \times X \\ \downarrow \pi_1 & & \\ \overline{\mathfrak{S}} \supseteq \mathfrak{S}^s & & \end{array}$$

where $Univ_{Quot}$ denotes the universal quotient sheaf over $Quot$. If we set $\pi_2: \overline{\mathfrak{S}} \times X \rightarrow X$ the projection to the second factor, one can consider a natural Kähler structure over the smooth points of \mathfrak{S}^s :

$$\Omega = (\pi_1)_*(\rho^* \omega_{FS, \Xi} \wedge \pi_2^* \omega_X^n),$$

where $\omega_{FS, \Xi}$ is the Fubini-Study metric on Ξ . The natural moment map associated to the $SL(\chi(k))$ action and with respect to Ω is given by

$$\mu(\mathcal{E}) = \int_X \mu_{FS, \Xi} \frac{\omega_X^n}{n!} \in Lie(SU(\chi(k)))^*,$$

and $\mu_{FS, \Xi}$ is the natural moment map on Ξ for the same action. Holomorphic vector bundles \mathcal{E} such that $\mu(\mathcal{E}) = 0$ correspond to Gieseker polystable bundle and for those bundles, there exists a “balanced” metric on $H^0(\mathcal{E} \otimes \mathcal{L}^k)$. This metric is actually a fixed point of a generalized T -map. Let us be more explicit. One can define two natural maps associated to the bundle case:

- The ‘Hilbertian’ map,

$$Hilb_k: \text{Met}(\mathcal{E} \otimes \mathcal{L}^k) \rightarrow \text{Met}(H^0(X, \mathcal{E} \otimes \mathcal{L}^k))$$

such that

$$Hilb_k(h)(s, \bar{s}) = \int_X |s|_h^2 \frac{\omega_L^n}{n!}.$$

- The ‘Fubini-Study’ map,

$$FS_k: \text{Met}(H^0(X, \mathcal{E} \otimes \mathcal{L}^k)) \rightarrow \text{Met}(\mathcal{E} \otimes \mathcal{L}^k)$$

such that

$$\sum_{i=1}^{\chi(k)} S_i \otimes S_i^{*FS_k(H)} = \frac{\chi(k)}{r \int_X c_1(\mathcal{L})^n} Id_{\mathcal{E} \otimes \mathcal{L}^k},$$

where S_i is an H -orthonormal basis of holomorphic sections of $H^0(\mathcal{E} \otimes \mathcal{L}^k)$.

Then, balanced metrics for the bundle \mathcal{E} correspond to fixed points of the map $T' = Hilb_k \circ FS_k$. If the bundle \mathcal{E} is Gieseker polystable, then T' admits a unique attractive fixed point. We are ready to define a notion of quantized Weil-Petersson metric for a family $\mathcal{E}_t \rightarrow X$ of Hermitian-Einstein vector bundles over $(X, \mathcal{L}, \omega_X)$ with fixed differential structure $E \rightarrow X$.

Definition 9. For $k \gg 0$ sufficiently large, let $H_t \in \text{Met}(H^0(X, \mathcal{E}_t \otimes \mathcal{L}^k))$ be the balanced metric of order k . For $v_1, v_2 \in \Omega^{0,1}(X, \text{End}(E))$, we define the following Kähler form,

$$\Omega_k(v_1, v_2) = k^n H_t^{\alpha, \beta} \int_X \langle \bar{\partial}^{-1}(v_1 \partial s_\alpha), \bar{\partial}^{-1}(v_2 \partial s_\beta) \rangle_{FS(H_t)} \frac{\omega_X^n}{n!},$$

for (s_α) an orthonormal basis with respect to H_t .

Note that the convergence of Ω_k is a consequence of the main theorem of [Wan2] and Section 4.1.2.

4.3. Numerical computation of the metrics. In [Don6], Donaldson showed how to numerically compute *balanced metrics* and ν -balanced metrics, in order to approximate cscK metrics on varieties and Kähler Ricci flat metrics on Calabi-Yau manifolds. Such metrics can be constructed explicitly if one has analytical control on the projective embeddings and knows how to evaluate integrals. The same technical difficulties arise if we want to evaluate quantized Weil-Petersson metrics on moduli spaces (4.7). For any basis of sections $\{s_\alpha\}_{\alpha=1}^{N+1}$, and definite positive Hermitian matrix $H_{\alpha\bar{\beta}}$ on the vector space $H^0(X_{t_0}, \mathcal{L}^k)$, one can introduce a L^2 -product on $\Omega^0(\mathcal{L}^k)$ defined as

$$(4.10) \quad \langle \sigma_1, \sigma_2 \rangle = \frac{1}{\text{vol}(X_{t_0})} \int_X \frac{\sigma_1 \bar{\sigma}_2}{(H^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \nu_{t_0} \wedge \bar{\nu}_{t_0},$$

with $\text{vol}(X_{t_0}) = \int_X \nu_{t_0} \wedge \bar{\nu}_{t_0}$, and $\sigma_1, \sigma_2 \in \Omega^0(\mathcal{L}^k)$. If the restriction of (4.10) on $H^0(X_{t_0}, \mathcal{L}^k) \subset L^2(X_{t_0}, \mathcal{L}^k)$ is identical to the inner product defined by H , one says

that the Fubini-Study metric defined by H on $\mathbb{P}H^0(X_{t_0}, \mathcal{L}^k)^*$ is ν -balanced² as we have seen in Section 4.1.3. One can find the balanced metric on $X_{t_0} \hookrightarrow \mathbb{P}^N$ by introducing any initial definite positive Hermitian matrix $H(0)$ on $H^0(X_{t_0}, \mathcal{L}^k)$, and iterating the map, $T: \text{Met}(H^0(\mathcal{L}^k)) \rightarrow \text{Met}(H^0(\mathcal{L}^k))$

$$(4.11) \quad H(a+1)_{\alpha\bar{\beta}} = T(H(a))_{\alpha\bar{\beta}} = \frac{N+1}{\text{vol}(X_{t_0})} \int_X \frac{s_\alpha \bar{s}_\beta}{(H(a)^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \nu_{t_0} \wedge \bar{\nu}_{t_0},$$

up to reaching convergence.

Let v denote an infinitesimal deformation on the moduli space, $v \in T_{t_0}\mathcal{T}$. In order to find the infinitesimal deformation of the balanced embedding for the manifold X_{t_0+v} into \mathbb{P}^N and be able to evaluate (4.7), it is convenient to work with a family of line bundles on X . If $\pi: \mathcal{X} \rightarrow \mathcal{T}$ denotes a family of complex structures on X , with $\pi^{-1}(t) = X_t$, there exists a holomorphic line bundle $\mathcal{S}^k \rightarrow \mathcal{X}$ such that $\mathcal{S}^k|_t = \mathcal{L}_t^k \rightarrow X_t$. In other words, the restriction of \mathcal{S}^k to the fibers of $\pi: \mathcal{X} \rightarrow \mathcal{T}$ is identical to the holomorphic polarization \mathcal{L}_t^k on X_t . The natural Hermitian structure on $\mathcal{L}_t^k \rightarrow X_t$ whose curvature is the corresponding Kähler Ricci flat metric, lifts to a Hermitian structure on $\mathcal{S}^k \rightarrow \mathcal{X}$. When we approximate the Kähler Ricci flat metrics on X_t by balanced metrics, $\mathcal{S}^k \rightarrow \mathcal{X}$ also admits a compatible Hermitian structure. More precisely, if $\{s_\alpha(t, \bar{t}) = \eta_\alpha(t, \bar{t})\hat{e}_t\}_{\alpha=1}^{N+1}$ is a basis of holomorphic sections for $H^0(X_t, \mathcal{L}^k)$, \hat{e}_t is the holomorphic frame in a local trivialization, the parameters t denote the moduli dependence, and $H(t, \bar{t})$ is the associated balanced Hermitian matrix, we endow $\mathcal{S}^k \rightarrow \mathcal{X}$ with the Hermitian metric

$$(4.12) \quad h_t = \frac{\hat{e}_t \otimes \hat{e}_t^*}{(H^{-1})^{\bar{\gamma}\delta}(t, \bar{t}) s(t)_\delta \bar{s}(\bar{t})_{\bar{\gamma}}}.$$

Therefore, given the diffeomorphism between X_{t_0} and X_{t_0+v} , defined in local holomorphic coordinate charts (3.1), as

$$y^i = w^i + v^a \vartheta_a^i(w, \bar{w}) + O(v^2),$$

one can compute the infinitesimal deformation of the embedding $X_{t_0+v} \hookrightarrow \mathbb{P}^N$, as the covariant derivative of $s_\alpha(t)$

$$(4.13) \quad \nabla_v \eta_\alpha \hat{e}_t = v^a \frac{\partial \eta_\alpha}{\partial t_a} \hat{e}_t + v^a h_t^{-1} \frac{\partial h_t}{\partial t_a} \eta_\alpha \hat{e}_t,$$

where $\frac{\partial \eta_\alpha}{\partial t_a} = \frac{\partial \eta_\alpha}{\partial w^i} \vartheta_a^i(w, \bar{w})$. In other words, if $\hat{e}(y)$ is a holomorphic frame for $\mathcal{L}_{t_0+tv}^k \rightarrow X_{t_0+v}$, we can write the basis of holomorphic sections as

$$(4.14) \quad s_\alpha(y) = \eta_\alpha(y) \hat{e}(y) = \eta_\alpha(w) \hat{e}(w) + \nabla_v \eta_\alpha(w, \bar{w}) \hat{e}(w) + O(v^2).$$

The proof is straightforward. Thus, $\nabla_v \eta_\alpha \hat{e}$ are smooth sections in $L^2(X_{t_0}, \mathcal{L}^k)$ that represent components of vector fields along $T^{1,0}|_{X_{t_0}} \mathbb{P}^N$. The sections $\nabla_v \eta_\alpha \hat{e} \in L^2(X_{t_0}, \mathcal{L}^k)$, can be expressed as the sum of a holomorphic section plus a non-holomorphic section, because of the decomposition

$$L^2(X_{t_0}, \mathcal{L}^k) = H^0(X_{t_0}, \mathcal{L}^k) \oplus H^0(X_{t_0}, \mathcal{L}^k)^\perp$$

under the L^2 -metric defined in (4.10). As we need the normal components of $\nabla_v \eta_\alpha \hat{e}$ to $H^0(X_{t_0}, \mathcal{L}^k)$, we have to project out the holomorphic part, $P_{t_0} \nabla_v \eta_\alpha \hat{e} \in$

²In this section, we will refer to ν -balanced metrics simply as “balanced metrics.”

$H^0(X_{t_0}, \mathcal{L}^k)$. The holomorphic part of $\nabla_v \eta_\alpha \hat{e}$ is computed using the Bergman kernel projector $P_{t_0} : L^2(X_{t_0}, \mathcal{L}^k) \rightarrow H^0(X_{t_0}, \mathcal{L}^k)$. In an orthonormal basis $\{s'_\alpha\}_{\alpha=1}^{N+1}$ one can express P_{t_0} as

$$(4.15) \quad P_{t_0} \sigma = \frac{N+1}{\text{vol}(X_{t_0})} \sum_{\alpha} s'_\alpha \int_X \frac{\sigma \bar{s}'^\alpha}{s'_\delta \bar{s}'^\delta} \nu_{t_0} \wedge \bar{\nu}_{t_0}.$$

Therefore, the term $(1 - P_{t_0}) \nabla_v \eta_\alpha \hat{e}$ denotes the projection of $\nabla_v \eta_\alpha \hat{e}$ onto the orthogonal complement in $\Gamma(T^{1,0}|_X \mathbb{P}^N)$ of the subspace defined by the infinitesimal action of $GL(N+1, \mathbb{C})$, which is isomorphic to $H^0(X_{t_0}, \mathcal{L}^k)$. Hence, the “quantized” Weil-Petersson metric (4.7) can be written as:

$$(4.16) \quad \Omega_k(v_1, v_2) = \frac{(H^{-1})^{\bar{\beta}\alpha}}{\text{vol}(X_{t_0})} \int_X \frac{((1 - P_{t_0}) \nabla_{v_1} \eta)_\alpha \overline{((1 - P_{t_0}) \nabla_{v_2} \eta)_{\bar{\beta}}}}{(H^{-1})^{\bar{\gamma}\delta} \eta_\delta \bar{\eta}_{\bar{\gamma}}} \nu_{t_0} \wedge \bar{\nu}_{t_0}.$$

Here, the basis of sections $\{s_\alpha = \eta_\alpha \hat{e}\}_{\alpha=1}^{N+1}$ is not necessarily orthonormal, although due to the simplicity of P_{t_0} when expressed in an orthonormal basis (4.15), it is convenient to work with $\{s_\alpha\}_{\alpha=1}^{N+1}$ orthonormal (where $H = \mathbf{1}$).

Thus, evaluating (4.16) involves:

- Building an explicit basis of sections $\{s_\alpha = \eta_\alpha \hat{e}\}_{\alpha=1}^{N+1}$ for $H^0(X_{t_0}, \mathcal{L}^k)$, with $k = 1, \dots, k_{max}$, and k_{max} some maximum value of k that one can handle numerically.
- Developing a numerical algorithm to evaluate integrals on X under the measure $\nu_{t_0} \wedge \bar{\nu}_{t_0}$, as we did in Section 3.1.
- Computing the balanced metric by iteration of the T-map, (4.11).
- Choosing a basis of infinitesimal diffeomorphisms $\vartheta_a^i(w, \bar{w})$ on X_{t_0} , isomorphic to the basis $\frac{\partial}{\partial t_a}$ for $T_{t_0} \mathcal{T} \simeq H^1(X_{t_0}, \Omega^{n-1})$.
- Solve the linearized balanced equations for $\partial_{t_a} H_{\alpha\bar{\beta}}$:

$$(4.17) \quad \begin{aligned} \frac{\partial H_{\alpha\bar{\beta}}}{\partial t_a} &= \xi_0 \int_X \frac{\nabla_a \eta_\alpha \hat{e} \bar{s}_{\bar{\beta}}}{(H^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \nu_{t_0} \wedge \bar{\nu}_{t_0} + \xi_0 \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{(H^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \frac{\partial \nu_t}{\partial t_a} \wedge \bar{\nu}_{t_0} \\ &\quad - \frac{\xi_0}{\text{vol}(X_{t_0})} \int_X \frac{\partial \nu_t}{\partial t_a} \wedge \bar{\nu}_{t_0} \times \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{(H^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \nu_{t_0} \wedge \bar{\nu}_{t_0}; \end{aligned}$$

where $\xi_0 = \frac{N+1}{\text{vol}(X_{t_0})}$. This is a non-trivial system of linear equations, as $\partial_{t_a} H_{\alpha\bar{\beta}}$ is contained in the $\nabla_a \eta_\alpha$ term. One can solve (4.17) by using Gauss’ elimination method, or one can solve it iteratively by using $\partial_{t_a} H_{\alpha\bar{\beta}}(0) = 0$ as initial value and interpreting Eq. (4.17) as a linearized T-map.

- Computing $\nabla_a \eta_\alpha$, given $\frac{\partial H_{\alpha\bar{\beta}}}{\partial t_a}$, and its projection $(1 - P_{t_0}) \nabla_a \eta_\alpha$, (4.15).
- Finally: Evaluating the inner products (4.16).

4.4. Example: the family of Quintics. We implemented the algorithm that we have just described, for the family of Quintic 3-folds Q_t in \mathbb{P}^4 defined by the polynomial

$$P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5t Z_0 Z_1 Z_2 Z_3 Z_4.$$

We studied the region of the t -plane given by $0 < |t| \leq 3$ and $0 \leq \arg(t) < 2\pi/5$, as we did in the examples of sections 2 and 3. We divided the region in a lattice of more than 300 points, and computed the corresponding balanced metrics for embeddings in linear spaces of sections, up to degree $k = 6$. We chose monomials of degree k defined on \mathbb{P}^4 , modulo the ideal generated by $P(Z)$, as the basis of

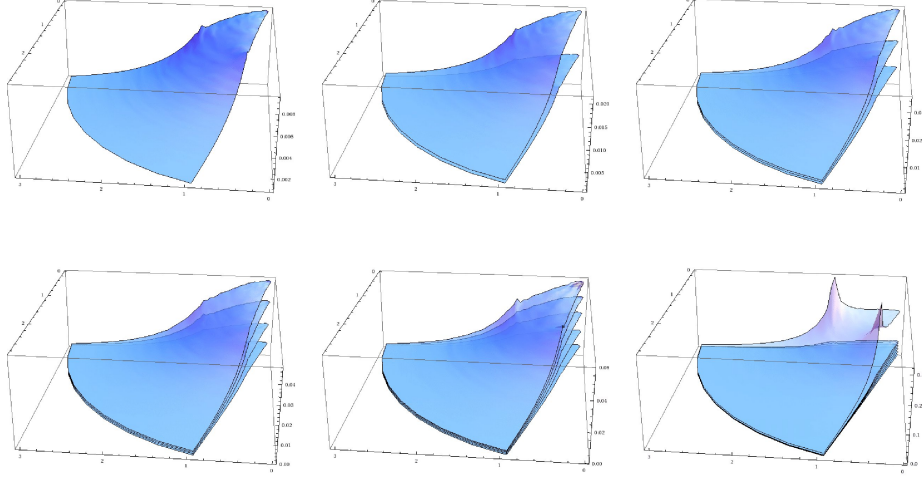


FIGURE 7. Kähler metrics (vertical axis) on the t -plane (horizontal plane), of Calabi-Yau Quintic threefolds $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4$, for $k = 1, 2, 3, 4, 5$, and 6 vs the exact Weil-Petersson metric.

sections. We evaluated the integrals that appear in the T-map (4.11) by using the Monte Carlo method described in section 3. In order to compute the variation of the sections $\frac{\partial s_\alpha}{\partial t}$, we used the infinitesimal diffeomorphism defined by (3.5), with

$$(4.18) \quad \frac{\partial \eta_\alpha}{\partial t} = \sum_{i=1}^4 \frac{\partial \eta_\alpha}{\partial w_i} \frac{\partial w_i}{\partial t} = \sum_{i=1}^4 \frac{\partial \eta_\alpha}{\partial w_i} \vartheta^i(w, \bar{w}).$$

For this family of Quintics, the equation (3.5) that defines our choice of vector field $\vartheta^i(w, \bar{w})$, becomes

$$(4.19) \quad \vartheta^i(w, \bar{w}) = - \frac{G^{i\bar{j}} \frac{\partial \bar{p}(\bar{w})}{\partial \bar{w}_j}}{G^{m\bar{n}} \frac{\partial \bar{p}(\bar{w})}{\partial \bar{w}_n} \frac{\partial p(w)}{\partial w_m}} (-5w_1w_2w_3w_4),$$

with $G^{i\bar{j}}$ the inverse of the Fubini-Study metric in \mathbb{P}^4 , and $w_i = Z_i/Z_0$ local coordinates on $Q_t \subset \mathbb{P}^4$. Given the Hermitian metric (4.12)

$$h_t = \frac{\hat{e}_t \otimes \hat{e}_t^*}{(H^{-1})^{\bar{\gamma}\delta}(t, \bar{t}) s(t)_\delta \bar{s}(\bar{t})_{\bar{\gamma}}},$$

we can compute the covariant derivative $\nabla_t \eta_\alpha \hat{e}$ as

$$\nabla_t \eta_\alpha \hat{e} = \sum_{i=1}^4 \frac{\partial \eta_\alpha}{\partial w_i} \vartheta^i(w, \bar{w}) \hat{e} - \frac{\partial}{\partial t} ((H^{-1})^{\bar{\gamma}\delta}(t, \bar{t}) \eta(t)_\delta \bar{\eta}(\bar{t})_{\bar{\gamma}}) \frac{\eta_\alpha \hat{e}}{(H^{-1})^{\bar{\gamma}\delta}(t, \bar{t}) \eta(t)_\delta \bar{\eta}(\bar{t})_{\bar{\gamma}}},$$

which we computed by using (4.18), and solving the linearized balanced equations (4.17) for $\frac{\partial H_{\alpha\bar{\beta}}}{\partial t}$. The method that we implemented to solve the linearized balanced equations, consisted in iterating the linearized T-map; such iterating scheme reached good estimates of the solutions within 5 or 6 iterations. In Fig. 7 we plot the sequence of metrics Ω_k , defined in (4.16), for $k = 1, 2, \dots, 6$. The time that took

to compute each value $\Omega_k(t, \bar{t})$ per point in the t -plane, was approximately equal to 4 times the time needed to compute the balanced metric. One can observe that for $|t|$ large, the rate of convergence of the sequence is higher than in other regions of the t -plane. In points near the Fermat Quintic, $t = 0$, and for $k = 6$, the quantized Kähler metric is approximately 0.07, and the exact value 0.19. One expects deviations smaller than 0.01 in this region of the t -plane, when $k > 12$. The worst rate of convergence is located near the points $t = \exp(2\sqrt{-1}\pi\mathbb{Z}/5)$, where Q develops double point singularities. In such region of the family, the approximation of the corresponding Ricci flat metric by ν -balanced metrics, is also much less accurate. One should develop further techniques to approximate accurately the metric near singular points of the moduli space.

One can explain intuitively why this scheme is not so accurate near singular points. The limit $k \gg 1$ corresponds to the semiclassical limit, in Kähler geometric quantization, of (X, ω) with Planck's constant $\hbar = \frac{\text{vol}(X_t)^{1/n}}{k}$. Due to quantum uncertainty in regions of volume smaller than \hbar^n , one expects that accurate approximations of geometric features in X occur when the size of such features is bigger than $\frac{\text{vol}(X_t)}{N+1} \simeq \frac{\text{vol}(X_t)}{k^n}$; i.e. the characteristic volume where such features are located is bigger than $\frac{\text{vol}(X_t)}{k^n}$. Therefore, as a singularity is a geometric object of zero volume, these numerical constructions should fail near singularities.

5. CONCLUSION

Weil-Petersson geometry of moduli spaces of polarized Calabi-Yau manifolds is a rich arena to study different methods that approximate Weil-Petersson metrics. In this paper we have introduced different algorithms and compared them with exact results. First, we have introduced a very fast algorithm to compute Weil-Petersson metrics by combining formulas for local deformations of the holomorphic form with Monte Carlo integration techniques. Also, by using the machinery of G.I.T., we have introduced a sequence of quantized Weil-Petersson metrics that converge to the Weil-Petersson metric. Building on ideas developed in Section 3, we have shown how these quantized metrics can be computed numerically. We have pointed out how one can use analogous ideas to compute metrics on moduli spaces of non-projective Calabi-Yau manifolds and moduli of stable vector bundles.

Another—more difficult and slower—way to approximate Weil-Petersson metrics involves to evaluate the Weil-Petersson formula itself on a family of balanced metrics; instead of a family of Kähler-Einstein metrics on varieties or Hermite-Einstein metrics on bundles. In other words, instead of using (4.16) to approximate Weil-Petersson metrics one could evaluate

$$(5.1) \quad \Upsilon_k(v_1, v_2) = \frac{1}{\text{vol}(X_t)} \int_X v_1^a \bar{v}_2^{\bar{b}} g_t^{i\bar{j}} \frac{\bar{\partial}}{\partial \bar{w}_{\bar{j}}} \left(h_t^{-1} \frac{\partial h_t}{\partial t_a} \right) \left(\frac{\bar{\partial}}{\partial \bar{w}_{\bar{i}}} \left(h_t^{-1} \frac{\partial h_t}{\partial t_b} \right) \right)^* \nu_t \wedge \bar{\nu}_t,$$

with h_t the family of balanced metrics defined in Eq. (4.12). As the formula (5.1) would become the Weil-Petersson metric if h_t was Hermite-Einstein, as in [ST], one expects that if h_t is balanced, Eq. (5.1) should converge to the Weil-Petersson metric in the $k \rightarrow \infty$ limit. We have implemented an algorithm to compute this metric on the family of Quintics that we have studied in this paper. As one can see in Eq. (5.1), it is slightly more difficult to implement this formula numerically due to the higher number of derivatives. Also, the numerical calculation itself is

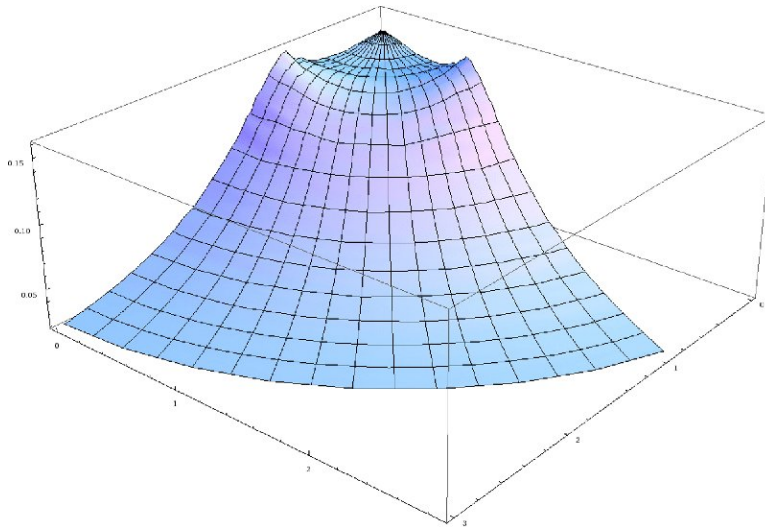


FIGURE 8. Evaluation of the Weil-Petersson formula (5.1) at a family of $k = 1$ balanced metrics on Calabi-Yau Quintic 3-folds, $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4$.

much slower in comparison with a numerical evaluation of (4.16). For instance, to compute the metric (5.1) for $k = 3$ took as much time as computing (4.16) for $k = 6$. Due to problems with the speed of the numerical calculation, we decided not to resume a detailed analysis of a numerical method based on Eq. (5.1).

Many other problems could be studied in further depth using our approach to compute numerical Weil-Petersson metrics. For instance, as an application, one could try to estimate the Weil-Petersson volume of the moduli space of Calabi-Yau quintics.

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CMI, 39 RUE FRÉDÉRIC JOLIOT-CURIE 13453 MARSEILLE, FRANCE.

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, 180 QUEEN'S GATE, LONDON, U.K.

E-mail address: `jkeller@cmi.univ-mrs.fr`, `slukic@imperial.ac.uk`